Relations

8.1. Introduction to Relations

8.2 Properties of Relations

8.3 Equivalence Relations

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Acknowledgement:
This lecture is based on (but not limited to) to chapter 5 in "Discrete Mathematics with Applications by Susanna S. Epp (3rd Edition)".

Relations

8.3 Equivalence Relations

In this lecture:

- Part 1: **Partitioned Sets**
- Part 2: **Equivalence Classes**
- Part 3: **Equivalence Relation**

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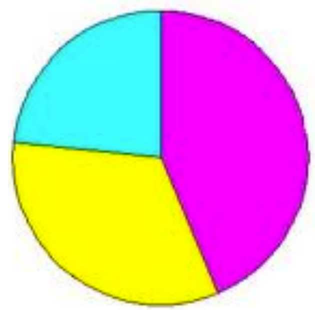
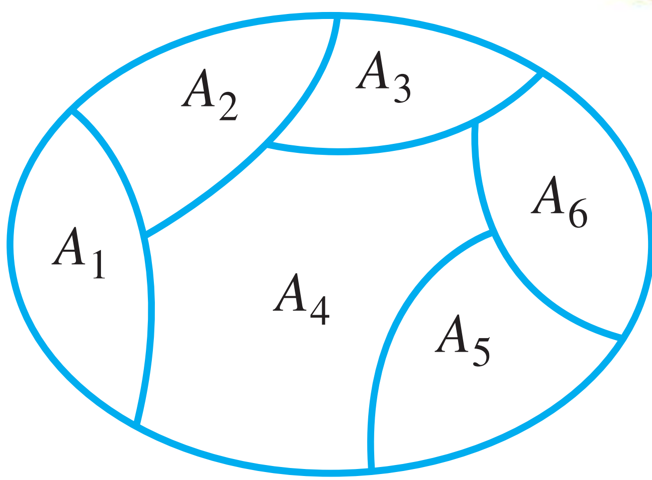
Partitioned Sets

Sets can be partitioned into disjoint sets

A **partition** of a set *A* is a finite or infinite collection of nonempty, mutually disjoint subsets whose union is *A*.





قسيم جامع مانع

Total (جامع)

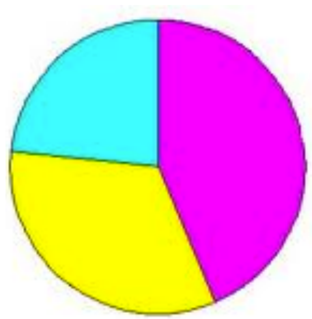
$$A_i \cup A_2 \cup \ldots \cup A_6 = A$$

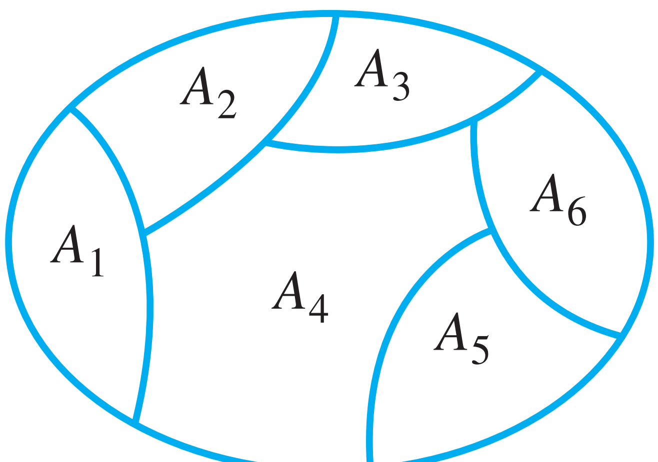
Disjoint (مانع)

$A_i \cap A_j = \phi$, whenever $i \neq j$

Partitioned Sets

Sets can be partitioned into disjoint sets





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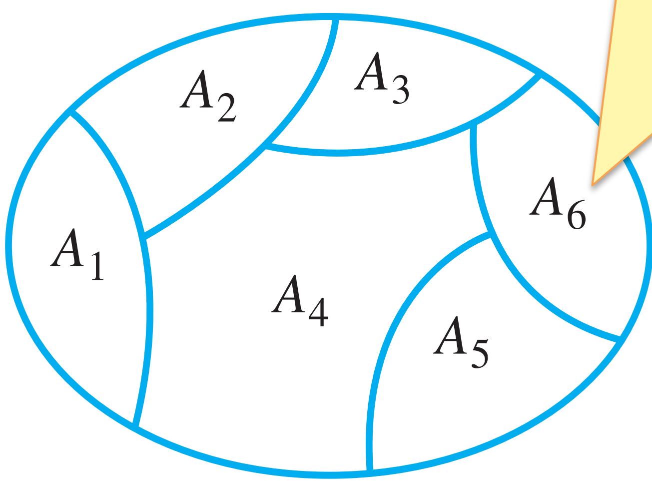
Relations Induced by a Partition

A **relation induced by a partition**, is a relation between two element in the same partition.



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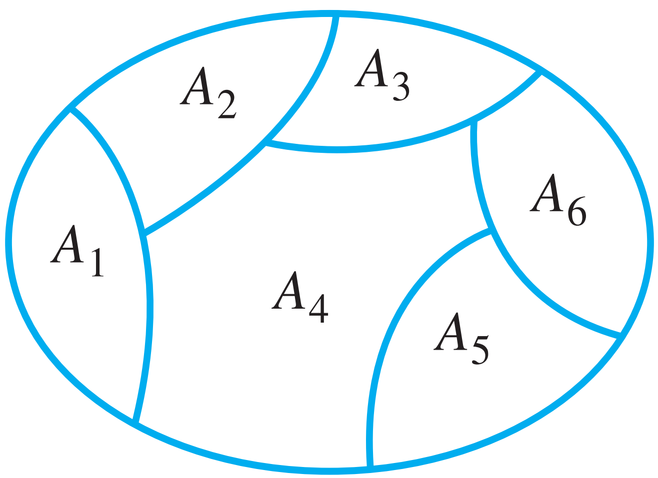
Relations Induced by a Partition

• Definition

Given a partition of a set A, the **relation induced by the partition,** R, is defined on A as follows: For all $x, y \in A$,

$$x \; R \; y \quad \Leftrightarrow \quad \text{there is a subset } A_i \text{ of the partition such that both } x \text{ and } y \text{ are in } A_i.$$



قسيم جامع مانع

Total (جامع)

$$A_i \cup A_2 \cup \ldots \cup \; A_6 = A$$

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$$A_i \cap A_j = \phi, \text{ whenever } i \neq j$$

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Example

Let $A = \{0, 1, 2, 3, 4\}$ and consider the following partition of A:

$\{0, 3, 4\}, \{1\}, \{2\}$.

Find the relation R induced by this partition.

Since {0, 3, 4} is a subset of the partition,

	$0\ R\ 3$	because both 0 and 3 are in {0, 3, 4},	
	$3\ R\ 0$	because both 3 and 0 are in {0, 3, 4},	
	$0\ R\ 4$	because both 0 and 4 are in {0, 3, 4},	
	$4\ R\ 0$	because both 4 and 0 are in {0, 3, 4},	
	$3\ R\ 4$	because both 3 and 4 are in {0, 3, 4},	and
	$4\ R\ 3$	because both 4 and 3 are in {0, 3, 4}.	
Also,	$0\ R\ 0$	because both 0 and 0 are in {0, 3, 4}	
	$3\ R\ 3$	because both 3 and 3 are in {0, 3, 4},	and
	$4\ R\ 4$	because both 4 and 4 are in {0, 3, 4}.	

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Example

Let $A = \{0, 1, 2, 3, 4\}$ and consider the following partition of A:

$$\{0, 3, 4\}, \{1\}, \{2\}.$$

Find the relation R induced by this partition.

Since {1} is a subset of the partition,

$1 \; R \; 1$ because both 1 and 1 are in {1},

and since {2} is a subset of the partition,

$2 \; R \; 2$ because both 2 and 2 are in {2}.

Hence

$$R = \{(0,0),(0,3),(0,4),(1,1),(2,2),(3,0),(3,3),(3,4),(4,0),(4,3),(4,4)\}.$$

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Relations Induced by a Partition

Theorem 8.3.1

Let A be a set with a partition and let R be the relation induced by the partition. Then R is reflexive, symmetric, and transitive.

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Relations

8.3 Equivalence Relations

In this lecture:

- ❑ Part 1: **Partitioned Sets**
- ❑ Part 2: **Equivalence Classes**
- ❑ Part 3: **Equivalence Relation**

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Equivalence Relation

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Definition

Let A be a set and R a relation on A. R is an **equivalence relation** if, and only if, R is reflexive, symmetric, and transitive.

➔ The relation induced by a partition is an equivalence relation

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Example

Let X be the set of all nonempty subsets of $\{1, 2, 3\}$. Then

$$X = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Define a relation R on X as follows: For all A and B in X,

$A\ R\ B \Leftrightarrow$ the least element of A equals the least element of B.

Prove that R is an equivalence relation on X.

R *is reflexive*: Suppose A is a nonempty subset of $\{1, 2, 3\}$. *[We must show that A* **R** *A.]* It is true to say that the least element of A equals the least element of A. Thus, by definition of R, A **R** A.

R *is symmetric*: Suppose A and B are nonempty subsets of $\{1, 2, 3\}$ and A **R** B. *[We must show that B* **R** *A.]* Since A **R** B, the least element of A equals the least element of B. But this implies that the least element of B equals the least element of A, and so, by definition of **R**, B **R** A.

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Example

Let X be the set of all nonempty subsets of $\{1, 2, 3\}$. Then

$$X = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Define a relation R on X as follows: For all A and B in X,

$$A\ R\ B \Leftrightarrow \text{the least element of } A \text{ equals the least element of } B.$$

Prove that R is an equivalence relation on X.

R *is transitive*: Suppose A, B, and C are nonempty subsets of $\{1, 2, 3\}$, A **R** B, and $B\ R\ C$. *[We must show that A* **R** *C.]* Since A **R** B, the least element of A equals the least element of B and since B **R** C, the least element of B equals the least element of C. Thus the least element of A equals the least element of C, and so, by definition of **R**, A **R** C.

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Example

Let S be the set of all digital circuits with a fixed number n of inputs. Define a relation E on S as follows: For all circuits $C1$ and $C2$ in S,

$$C_1 \mathbf{E}\, C_2 \Leftrightarrow C1 \text{ has the same input/output table as } C2.$$

E *is reflexive*: Suppose C is a digital logic circuit in S. *[We must show that C* **E** *C.]* Certainly C has the same input/output table as itself. Thus, by definition of **E**, C **E** C

E *is symmetric*: Suppose C_1 and C_2 are digital logic circuits in S such that C_1 **E** C_2. By definition of **E**, since C_1 **E** C_2, then C_1 has the same input/output table as C_2. It follows that C_2 has the same input/output table as C_1. Hence, by definition of **E**, C_2 **E** C_1

E *is transitive*: Suppose C_1, C_2, and C_3 are digital logic circuits in S such that C_1 **E** C_2 and C_2 **E** C_3. By definition of **E**, since C_1 **E** C_2 and C_2 **E** C_3, then C_1 has the same input/output table as C_2 and C_2 has the same input/output table as C_3. It follows that C_1 has the same input/output table as C_3. Hence, by definition of **E**, C_1 **E** C_3

Example

Let L be the set of all allowable identifiers in a certain computer language, and define a relation R on L as follows: For all strings s and t in L,

$s\ R\ t \Leftrightarrow$ the first eight characters of s equal the first eight characters of t.

***R is reflexive*:** Let $s \in L$. Clearly s has the same first eight characters as itself. Thus, by definition of R, $s\ R\ s$.

***R is symmetric*:** Let s and t be in L and suppose that $s\ R\ t$. By definition of R, since $s\ R\ t$, the first eight characters of s equal the first eight characters of t. But then the first eight characters of t equal the first eight characters of s. And so, by definition of R, $t\ R\ s$

***R is transitive*:** Let s, t, and u be in L and suppose that $s\ R\ t$ and $t\ R\ u$. By definition of R, since $s\ R\ t$ and $t\ R\ u$, the first eight characters of s equal the first eight characters of t, and the first eight characters of t equal the first eight characters of u. Hence the first eight characters of s equal the first eight characters of u. Thus, by definition of R, $s\ R\ u$

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8.3 Equivalence Relations

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Equivalence Class

• Definition

Suppose A is a set and R is an equivalence relation on A. For each element a in A, the **equivalence class of a,** denoted **$[a]$** and called the **class of a** for short, is the set of all elements x in A such that x is related to a by R.

In symbols:

$$[a] = \{x \in A \mid x \; R \; a\}$$

$$\text{for all } x \in A, \quad x \in [a] \quad \Leftrightarrow \quad x \; R \; a.$$

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Example

Let $A = \{0,1,2,3,4\}$ and define a relation R on A as :
$R = \{(0,0), (0,4), (1,1), (1,3), (2,2), (3,1), (3,3), (4,0), (4,4)\}$.

Find the distinct equivalence classes of R.

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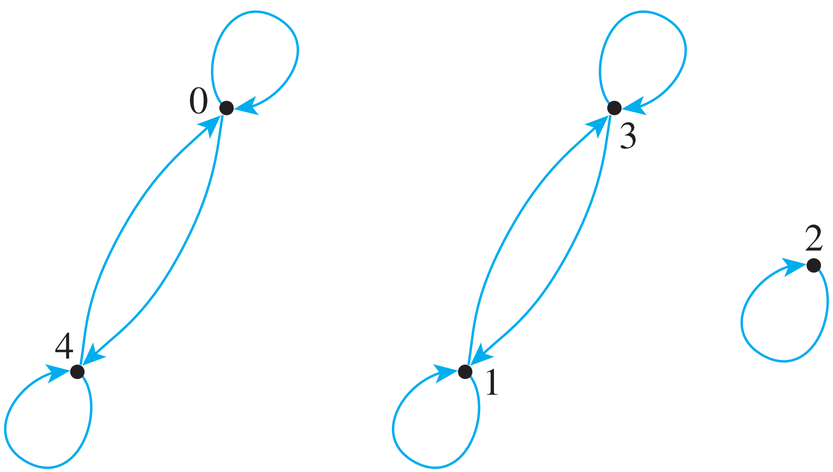
Example

Let $A = \{0,1,2,3,4\}$ and define a relation R on A as :
$R = \{(0, 0), (0, 4), (1, 1), (1, 3), (2, 2), (3, 1), (3, 3), (4, 0), (4, 4)\}$.

Find the distinct equivalence classes of R.



$$[0] = \{x \in A \mid x \ R \ 0\} = \{0, 4\}$$
$$[1] = \{x \in A \mid x \ R \ 1\} = \{1, 3\}$$
$$[2] = \{x \in A \mid x \ R \ 2\} = \{2\}$$
$$[3] = \{x \in A \mid x \ R \ 3\} = \{1, 3\}$$
$$[4] = \{x \in A \mid x \ R \ 4\} = \{0, 4\}$$

$[0] = [4]$ and $[1] = [3]$. Thus the *distinct* equivalence classes of the relation are $\{0, 4\}$, $\{1, 3\}$, and $\{2\}$.

Equivalence Class

Lemma 8.3.2

Suppose A is a set, R is an equivalence relation on A, and a and b are elements of A. If $a \ R \ b$, then $[a] = [b]$.

Lemma 8.3.3

If A is a set, R is an equivalence relation on A, and a and b are elements of A, then

$$\text{either} \quad [a] \cap [b] = \emptyset \quad \text{or} \quad [a] = [b].$$

• Definition

Suppose R is an equivalence relation on a set A and S is an equivalence class of R. A **representative** of the class S is any element a such that $[a] = S$.

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Congruence Modulo 3

Let R be the relation of congruence modulo 3 on the set $\mathbf{Z}$ of all integers. That is, for all integers m and n,

$$m\ R\ n \Leftrightarrow 3|(m-n) \Leftrightarrow m \equiv n \pmod 3.$$

Describe the distinct equivalence classes of R.

For each integer a,

$$\begin{aligned}[a] &= \{x \in \mathbf{Z} \mid x\ R\ a\} \\ &= \{x \in \mathbf{Z} \mid 3 \mid (x-a)\} \\ &= \{x \in \mathbf{Z} \mid x - a = 3k, \text{ for some integer } k\}.\end{aligned}$$

Therefore

$$[a] = \{x \in \mathbf{Z} \mid x = 3k + a, \text{ for some integer } k\}.$$

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Congruence Modulo 3

Let R be the relation of congruence modulo 3 on the set $\mathbf{Z}$ of all integers. That is, for all integers m and n,

$$mRn \Leftrightarrow 3|(m-n) \Leftrightarrow m \equiv n (\text{mod} 3).$$

In particular:

$$\begin{aligned}
[0] &= \{x \in \mathbf{Z} \mid x = 3k + 0, \text{ for some integer } k\} \\
&= \{x \in \mathbf{Z} \mid x = 3k, \text{ for some integer } k\} \\
&= \{\ldots -9, -6, -3, 0, 3, 6, 9, \ldots\}, \\
[1] &= \{x \in \mathbf{Z} \mid x = 3k + 1, \text{ for some integer } k\} \\
&= \{\ldots -8, -5, -2, 1, 4, 7, 10, \ldots\}, \\
[2] &= \{x \in \mathbf{Z} \mid x = 3k + 2, \text{ for some integer } k\} \\
&= \{\ldots -7, -4, -1, 2, 5, 8, 11, \ldots\}.
\end{aligned}$$

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Congruence Modulo 3

Let R be the relation of congruence modulo 3 on the set **Z** of all integers. That is, for all integers m and n,

$$mRn \Leftrightarrow 3|(m-n) \Leftrightarrow m \equiv n \pmod 3.$$

Now since $3\ R\ 0$, then by Lemma 8.3.2,

$$[3] = [0].$$

More generally, by the same reasoning,

$$[0] = [3] = [-3] = [6] = [-6] = \ldots, \text{ and so on.}$$

Similarly,

$$[1] = [4] = [-2] = [7] = [-5] = \ldots, \text{ and so on.}$$

And

$$[2] = [5] = [-1] = [8] = [-4] = \ldots, \text{ and so on.}$$

Congruence Modulo 3

Let R be the relation of congruence modulo 3 on the set $\mathbf{Z}$ of all integers. That is, for all integers m and n,

$$mRn \Leftrightarrow 3|(m-n) \Leftrightarrow m \equiv n(\text{mod}3).$$

Notice that every integer is in class [0], [1], or [2]. Hence the distinct equivalence classes are

$$\{x \in \mathbf{Z} \mid x = 3k, \text{ for some integer } k\},$$

$$\{x \in \mathbf{Z} \mid x = 3k + 1, \text{ for some integer } k\},$$

$$\{x \in \mathbf{Z} \mid x = 3k + 2, \text{ for some integer } k\}.$$

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Congruence Modulo 3

Determine which of the following congruences are true and which are false.

a. $12 \equiv 7 (\mathrm{mod} 5)$ b. $6 \equiv -8 (\mathrm{mod} 4)$ c. $3 \equiv 3 (\mathrm{mod} 7)$

a. True. $12 - 7 = 5 = 5 \cdot 1$. Hence $5 \mid (12 - 7)$, and so $12 \equiv 7 \pmod 5$.

b. False. $6 - (-8) = 14$, and $4 \nmid 14$ because $14 \neq 4 \cdot k$ for any integer k. Consequently, $6 \not\equiv -8 \pmod 4$.

c. True. $3 - 3 = 0 = 7 \cdot 0$. Hence $7 \mid (3 - 3)$, and so $3 \equiv 3 \pmod 7$. ■

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Exercise

Let A be the set of all ordered pairs of integers for which the second element of the pair is nonzero. Symbolically,

$$A = \mathbf{Z} \times (\mathbf{Z} - \{0\}).$$

Define a relation R on A as follows: For all $(a, b), (c, d) \in A$,

$$(a,b) R (c,d) \Leftrightarrow ad = bc.$$

Describe the distinct equivalence classes of R

For example, the class (1,2):

$$[(1, 2)] = \{(1, 2), (-1, -2), (2, 4), (-2, -4), (3, 6), (-3, -6), \ldots\}$$

since $\frac{1}{2} = \frac{-1}{-2} = \frac{2}{4} = \frac{-2}{-4} = \frac{3}{6} = \frac{-3}{-6}$ and so forth.

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