Mustafa Jarrar: Lecture Notes in Discrete Mathematics.
Birzeit University, Palestine, 2015

Counting

9.1 Basics of Probability and Counting

9.2 Possibility Trees and the Multiplication Rule

9.3 Counting Elements of Disjoint Sets: Addition Rule

9.5 Counting Subsets of a Set: Combinations

9.6 r-Combinations with Repetition Allowed

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Watch this lecture and download the slides

http://jarrar-courses.blogspot.com/2014/03/discrete-mathematics-course.html

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Acknowledgement:
This lecture is based on (but not limited to) to chapter 5 in "Discrete Mathematics with Applications by Susanna S. Epp (3rd Edition)".

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9.2 Possibility Trees and the Multiplication Rule

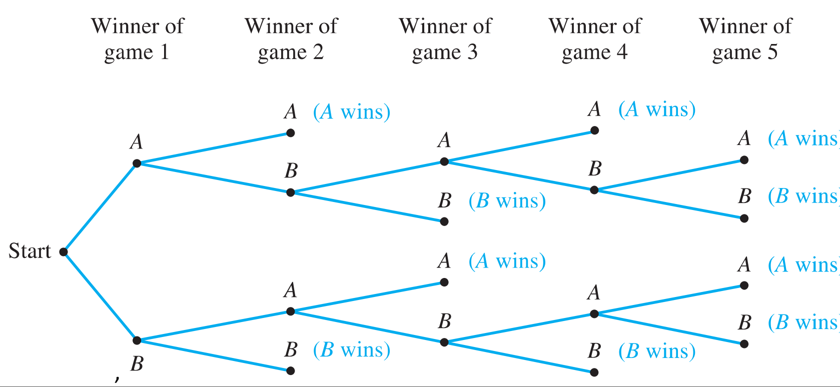
In this lecture:

- ❑ Part 1: **Possibility Trees**
- ❑ Part 2: **Multiplication Rule**
- ❑ Part 3: **Permutations**

Possibility Trees

Teams *A* and *B* are to play each other repeatedly until one wins two games in a row or a total of three games

a- How many ways can the tournament be played?
b- Assuming that all the ways of playing the tournament are equally likely, what is the probability that five games are needed to determine the tournament winner?



Possibility Trees

Teams A and B are to play each other repeatedly until one wins two games in a row or a total of three games

a. A–A, A–B–A–A, A–B–A–B–A, A–B–A–B–B, A–B–B, B–A–A, B–A–B–A–A, B–A–B–A–B, B–A–B–B, and B–B.
* In five cases A wins, and in the other five B wins.

b. *Since all the possible ways of playing the tournament listed in part (a) are assumed to be equally likely, and the listing shows that five games are needed in four different cases (A–B–A–B–A, A–B–A–B–B, B–A–B–A–B, and B–A–B–A–A), the probability that five games are needed is 4/10 = 2/5 = 40%.*

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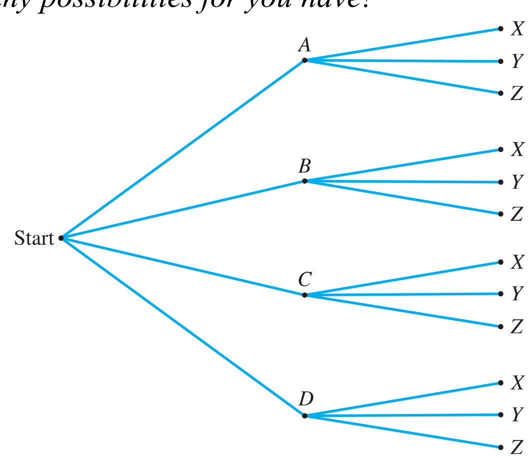
Possibility Trees

We have 4 computers (A,B,C,D) and 3 printers (X,Y,Z). Each of these printers is connected with each of the computers. *Suppose you want to print something through one of the computers, How many possibilities for you have?*



$3+3+3+3 = 4 \cdot 3 = 12.$

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Possibility Trees

A person buying a personal computer system is offered a choice of three models of the basic unit, two models of keyboard, and two models of printer.
How many distinct systems can be purchased?

Possibility Trees

Notices that representing the possibilities in a tree structure is a useful tool for tracking all possibilities in situations in which events happen in order.

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The Multiplication Rule

Theorem 9.2.1 The Multiplication Rule

If an operation consists of k steps and

the first step can be performed in n_1 ways,

the second step can be performed in n_2 ways *[regardless of how the first step was performed]*,

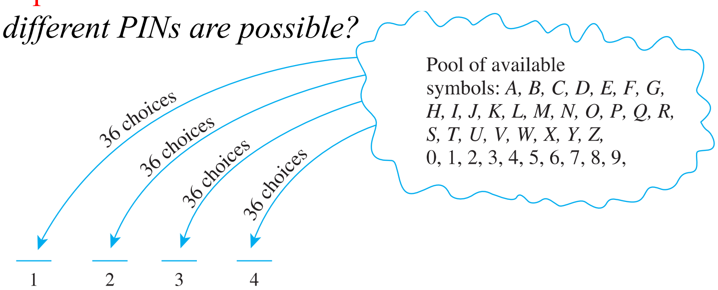
$\vdots$

the kth step can be performed in n_k ways *[regardless of how the preceding steps were performed]*,

then the entire operation can be performed in $n_1 n_2 \cdots n_k$ ways.

Counting Example 1

A typical PIN (personal identification number) is a sequence of any four symbols chosen from the 26 letters in the alphabet and the 10 digits, with repetition allowed.
How many different PINs are possible?



Step 1: Choose the first symbol.
Step 2: Choose the second symbol.
Step 3: Choose the third symbol.
Step 4: Choose the fourth symbol.

$36 \cdot 36 \cdot 36 \cdot 36 = 36^4 = 1{,}679{,}616$ PINs in all.

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Counting Example 1

A typical PIN (personal identification number) is a sequence of any four symbols chosen from the 26 letters in the alphabet and the 10 digits, with repetition **not** allowed.
How many different PINs are possible?

$$36 \cdot 35 \cdot 34 \cdot 33 = 1{,}413{,}720$$

what is the probability that a PIN chosen at random contains no repeated symbol?

$$\frac{1{,}413{,}720}{1{,}679{,}616} \cong .8417$$

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Counting Example 2

for $i := 1$ **to** 4
  for $j := 1$ **to** 3
    [*Statements in body of inner loop. None contain branching statements that lead out of the inner loop.*]
  next j
next i

How many times this statement will be executed?

Counting Example 3

Suppose A_1, A_2, A_3, and A_4 are sets with n_1, n_2, n_3, and n_4 elements, respectively.

How many elements in $A_1 \times A_2 \times A_3 \times A_4$

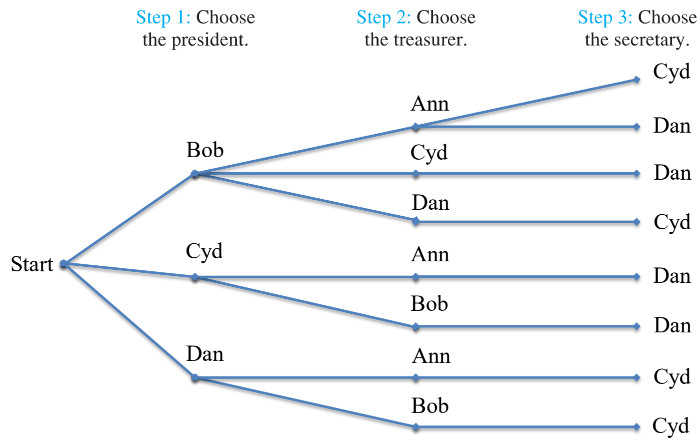
Solution: Each element in $A_1 \times A_2 \times A_3 \times A_4$ is an ordered 4-tuple of the form (a_1,a_2,a_3,a_4)

By the multiplication rule, there are $n_1n_2n_3n_4$ ways to perform the entire operation. Therefore, there are $n_1n_2n_3n_4$ distinct 4-tuples in $A_1 \times A_2 \times A_3 \times A_4$

Counting Example 4

Three officers—a president, a treasurer, and a secretary—are to be chosen from among four people: Ann, Bob, Cyd, and Dan. Suppose that, Ann cannot be president and either Cyd or Dan must be secretary. How many ways can the officers be chosen?

$3 \cdot 3 \cdot 2 = 18$



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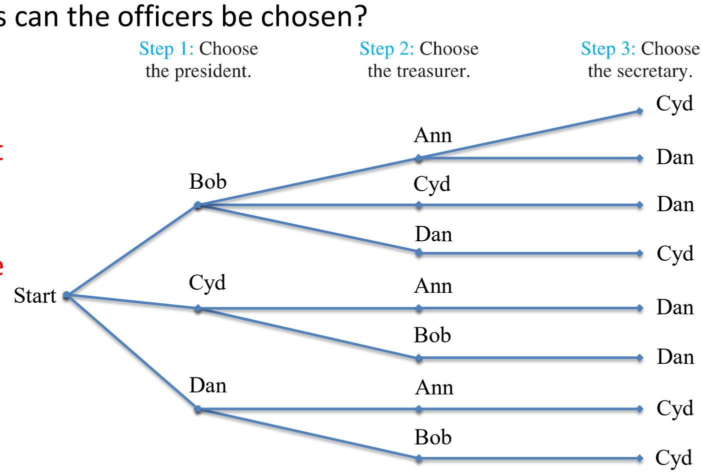
Counting Example 4

Three officers—a president, a treasurer, and a secretary—are to be chosen from among four people: Ann, Bob, Cyd, and Dan. Suppose that, Ann cannot be president and either Cyd or Dan must be secretary. How many ways can the officers be chosen?

This tree is not homogenous, thus we cannot use the multiplication rule!!

Better Idea?



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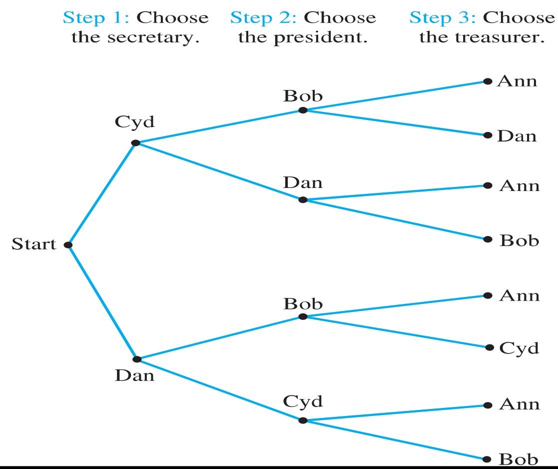
Counting Example 4—reorder the steps to get the correct number of ways by the multiplication rule

تم انتخاب أربعة طلاب لنادي الكلية: (Ann, Bob, Cyd, Dan). نريد اختيار رئيس، أمين صندوق، وسكرتير. لا يمكن ل Ann ان تكون رئيساً، والسكرتير اما ان يكون Dan او Cyd كم تشكيلة ممكنة؟

2.2.2 = 8

We should be smart to represent our problem in a way to be able to use the multiplication rule



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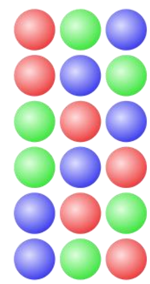
Permutations



التباديل : عدد التشكيلات الممكنة لمجموعة جزئية من العناصر منتقاة من مجموعة كلية من العناصر مع مراعاة لأهمية تسلسل العناصر في تشكيلات المجموعة الجزئية

كم كلمة من خمس حروف ممكن ان نكون اذا كان لدينا عشرة حروف؟

كانت القاعدة التي تمكن من حساب عدد التبديلات لمجموعة ما، معروفة لدى الهنديين على الأقل في حوالي عام 1150م.

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Permutations

A **permutation** of a set of objects is an ordering of the objects in a row.

For example, the set of elements $\{a, b, c\}$ has six permutations.

$$abc \quad acb \quad cba \quad bac \quad bca \quad cab$$

Generally, given a set of n objects, how many permutations does the set have? Imagine forming a permutation as an n-step operation:

Step 1: Choose an element to write **first**.
Step 2: Choose an element to write **second**
. . .
Step *n*: Choose an element to write ***n*th**.

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Permutations

by the multiplication rule, there are

$$n(n-1)(n-2)\cdot\cdot\cdot 2\cdot 1 = n!$$

ways to perform the entire operation.

Theorem 9.2.2

For any integer n with $n \geq 1$, the number of permutations of a set with n elements is $n!$.

Example 1

How many ways can the letters in the word *COMPUTER* be arranged in a row?

8! = 40,320

How many ways can the letters in the word *COMPUTER* be arranged if the letters *CO* must remain next to each other (in order) as a unit?

7! = 5,040

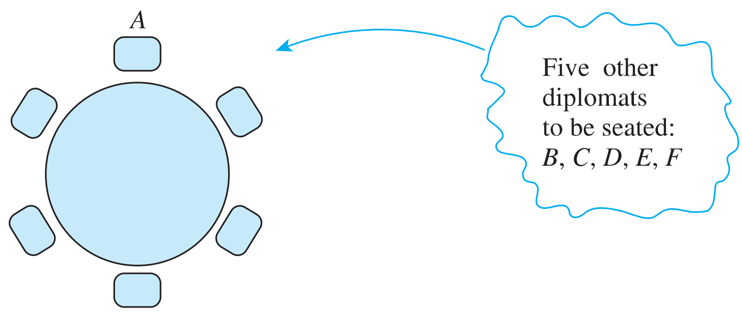
If letters of the word *COMPUTER* are randomly arranged in a row, what is the probability that the letters *CO* remain next to each other (in order) as a unit?

When the letters are arranged randomly in a row, the total number of arrangements is 40,320 by part (a), and the number of arrangements with the letters *CO* next to each other (in order) as a unit is 5,040.

$$\frac{5{,}040}{40{,}320} = \frac{1}{8} = 12.5\%.$$

Example 2

كيف يمكن توزيع ستة دبلوماسيين حول طاولة مستديرة



لانها مستديرة، ثبت واحدة، ويبقى خمسة يمكن تبديلها

Permutations of Selected Elements

Given the set $\{a, b, c\}$, there are six ways to select two letters from the set and write them in order.

$$ab \quad ac \quad ba \quad bc \quad ca \quad cb$$

Each such ordering of two elements of $\{a, b, c\}$ is called a 2-*permutation* of $\{a, b, c\}$.

أي مجموع التبديلات التي يمكن أن ننتقي بها أفراد المجموعة مع مراعاة الترتيب

• Definition

An ***r*-permutation** of a set of n elements is an ordered selection of r elements taken from the set of n elements. The number of r-permutations of a set of n elements is denoted $\boldsymbol{P(n, r)}$.

How many permutations in $\boldsymbol{P(n, r)}$?

Permutations of Selected Elements

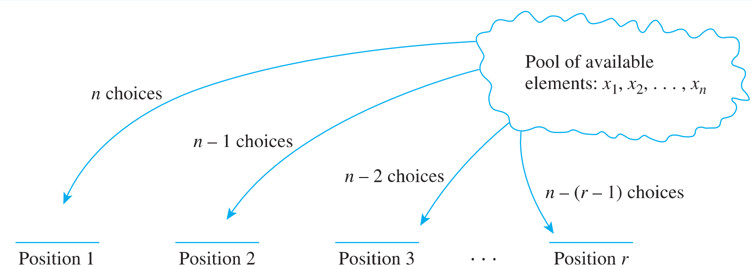
Theorem 9.2.3

If n and r are integers and $1 \leq r \leq n$, then the number of r-permutations of a set of n elements is given by the formula

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1) \qquad \text{first version}$$

or, equivalently,

$$P(n, r) = \frac{n!}{(n-r)!} \qquad \text{second version.}$$



Example 3

a. Evaluate $P(5, 2)$.

$$P(5, 2) = \frac{5!}{(5-2)!} = \frac{5\cdot 4\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}{\cancel{3}\cdot \cancel{2}\cdot \cancel{1}} = 20$$

b. How many 4-permutations are there of a set of 7 objects?

$$P(7, 4) = \frac{7!}{(7-4)!} = \frac{7\cdot 6\cdot 5\cdot 4\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}{\cancel{3}\cdot \cancel{2}\cdot \cancel{1}} = 7\cdot 6\cdot 5\cdot 4 = 840.$$

c. How many 5-permutations are there of a set of 5 objects?

$$P(5, 5) = \frac{5!}{(5-5)!} = \frac{5!}{0!} = \frac{5!}{1} = 5! = 120.$$

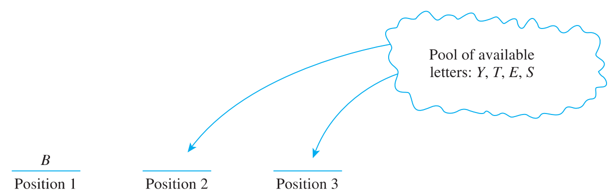
Example 4

How many different ways can 3 of the letters of the word *BYTES* be chosen and written in a row?

$$P(5, 3) = \frac{5!}{(5-3)!} = \frac{5 \cdot 4 \cdot 3 \cdot \cancel{2} \cdot \cancel{1}}{\cancel{2} \cdot \cancel{1}} = 5 \cdot 4 \cdot 3 = 60.$$

How many different ways can this be done if the first letter must be *B*?

$$P(4, 2) = \frac{4!}{(4-2)!} = \frac{4 \cdot 3 \cdot \cancel{2} \cdot \cancel{1}}{\cancel{2} \cdot \cancel{1}} = 4 \cdot 3 = 12.$$



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Example 5

Prove that for all integers $n \geq 2$,

$$P(n, 2) + P(n, 1) = n^2.$$

$$P(n, 2) = \frac{n!}{(n-2)!} = \frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}} = n(n-1)$$

and

$$P(n, 1) = \frac{n!}{(n-1)!} = \frac{n \cdot \cancel{(n-1)!}}{\cancel{(n-1)!}} = n.$$

Hence

$$P(n, 2) + P(n, 1) = n \cdot (n-1) + n = n^2 - n + n = n^2,$$

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