Mustafa Jarrar: Lecture Notes in Discrete Mathematics.
Birzeit University, Palestine, 2015

Counting

9.1 Basics of Probability and Counting

9.2 Possibility Trees and the Multiplication Rule

9.3 Counting Elements of Disjoint Sets: Addition Rule

9.5 Counting Subsets of a Set: Combinations

9.6 r-Combinations with Repetition Allowed

1

Watch this lecture
and download the slides

http://jarrar-courses.blogspot.com/2014/03/discrete-mathematics-course.html

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Acknowledgement:
This lecture is based on (but not limited to) to chapter 5 in "Discrete Mathematics with Applications by Susanna S. Epp (3rd Edition)".

2

Counting

9.3 Counting Elements of Disjoint Sets: Addition Rule

In this lecture:

❑ Part 1: **Addition Rule**

❑ Part 2: **Difference Rule**

❑ Part 2: **Inclusion Rule**

Apply these rules to count elements of union and disjoint sets

3

Additional Rule

e.g., Number of students in this class = Number of Girls + Number of boys, in this class

Theorem 9.3.1 The Addition Rule

Suppose a finite set A equals the union of k distinct mutually disjoint subsets A_1, $A_2, \ldots, A_k$. Then
$$N(A) = N(A_1) + N(A_2) + \cdots + N(A_k).$$

The number of elements in a union of mutually disjoint finite sets equals the sum of the number of elements in each of the component sets.

4

Exercise

A password consists of from 1, 2, or 3 letters chosen from {a..z} with repetitions allowed. How many different passwords are possible?

passwords of length 1	passwords of length 2	passwords of length 3

5

Exercise

A password consists of from 1, 2, or 3 letters chosen from {a..z} with repetitions allowed. How many different passwords are possible?
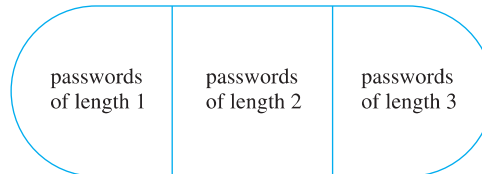
passwords of length 1	passwords of length 2	passwords of length 3
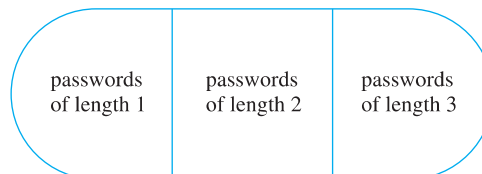
Number of passwords of length $1 = 26$ (because there are 26 letters in the alphabet)

Number of passwords of length $2 = 26^2$ (two-step process in which there are 26 ways to perform each step)

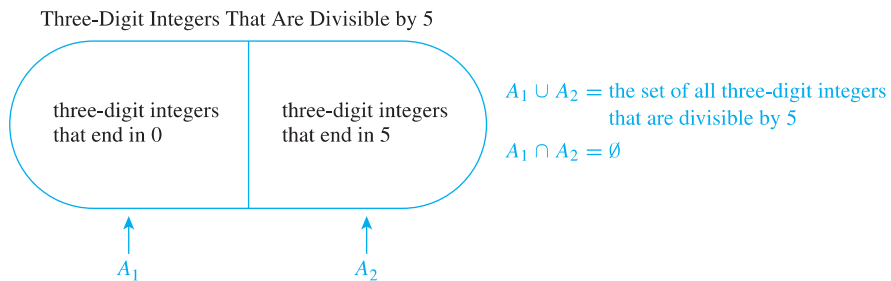Number of passwords of length $3 = 26^3$

Total $= 26 + 26^2 + 26^3 = 18{,}278.$

6

3

Exercise

How many three-digit integers (i.e., integers from 100 to 999 inclusive) are divisible by 5? (using the addition rule)
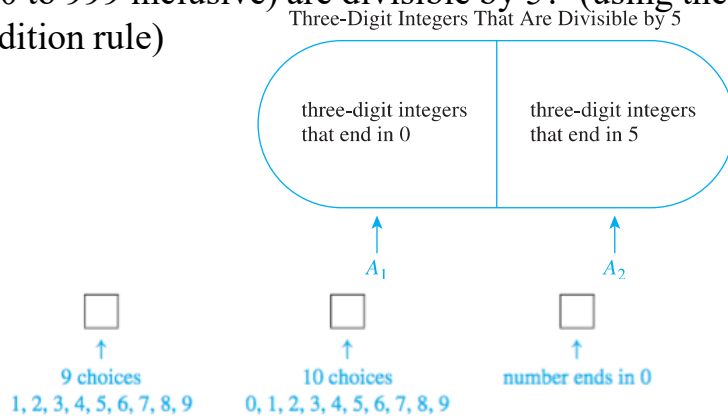
Integers that are divisible by 5 end either in 5 or in 0. Thus the set of all three-digit integers that are divisible by 5 can be split into two mutually disjoint subsets A_1 and A_2

Three-Digit Integers That Are Divisible by 5

three-digit integers that end in 0

three-digit integers that end in 5

$A_1 \cup A_2 =$ the set of all three-digit integers that are divisible by 5

$A_1 \cap A_2 = \emptyset$

A_1

A_2

7

Exercise

How many three-digit integers (i.e., integers from 100 to 999 inclusive) are divisible by 5? (using the addition rule)

Three-Digit Integers That Are Divisible by 5

three-digit integers that end in 0

three-digit integers that end in 5

A_1

A_2

9 choices
1, 2, 3, 4, 5, 6, 7, 8, 9

10 choices
0, 1, 2, 3, 4, 5, 6, 7, 8, 9

number ends in 0

$$\begin{bmatrix} \text{the number of} \\ \text{three-digit integers} \\ \text{that are divisible by 5} \end{bmatrix} = N(A_1) + N(A_2) = 90 + 90 = 180.$$

8

Counting

9.3 Counting Elements of Disjoint Sets: Addition Rule

In this lecture:

❑ Part 1: **Addition Rule**

❑ Part 2: **Difference Rule**

❑ Part 2: **Inclusion Rule**

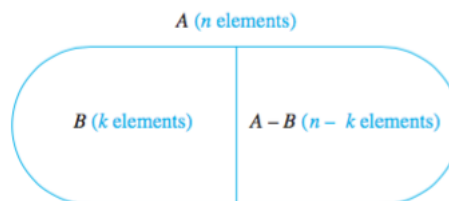Apply these rules to count elements of union and disjoint sets

9

The Difference Rule

Number of students without girls =
number of all students – number of girls

Theorem 9.3.2 The Difference Rule

If A is a finite set and B is a subset of A, then

$$N(A - B) = N(A) - N(B).$$

A (n elements)

B (k elements) | A − B ($n - k$ elements)

10

Exercise

The PIN codes are made from exactly four symbols chosen from the 26 letters and the 10 digits, with repetitions allowed.

a) How many PINs contain repeated symbols?

$$1{,}679{,}616 - 1{,}413{,}720 = 265{,}896$$

Exercise

The PIN codes are made from exactly four symbols chosen from the 26 letters and the 10 digits, with repetitions allowed.

➢ **If all PINs are equally likely, what is the probability that a randomly chosen PIN contains a repeated symbol?**

One way

$$\frac{265{,}896}{1{,}679{,}616} \cong 0.158 = 15.8\%.$$

Another way

$$P(S - A) = \frac{N(S - A)}{N(S)} \quad \text{by definition of probability in the equally likely case}$$

$$= \frac{N(S) - N(A)}{N(S)} \quad \text{by the difference rule}$$

$$= \frac{N(S)}{N(S)} - \frac{N(A)}{N(S)} \quad \text{by the laws of fractions}$$

$$= 1 - P(A) \quad \text{by definition of probability in the equally likely case}$$

$$\cong 1 - 0.842 \quad \text{by Example 9.2.4}$$

$$\cong 0.158 = 15.8\%$$

Exercise

The PIN codes are made from exactly four symbols chosen from the 26 letters and the 10 digits, with repetitions allowed.

➤ **If all PINs are equally likely, what is the probability that a randomly chosen PIN contains a repeated symbol?**

Formula for the Probability of the Complement of an Event

If S is a finite sample space and A is an event in S, then

$$P(A^c) = 1 - P(A).$$

Another way

$$= \frac{N(S) - N(A)}{N(S)} \quad \text{by the difference rule}$$

$$= \frac{N(S)}{N(S)} - \frac{N(A)}{N(S)} \quad \text{by the laws of fractions}$$

$$= 1 - P(A) \quad \text{by definition of probability in the equally likely case}$$

$$\cong 1 - 0.842 \quad \text{by Example 9.2.4}$$

$$\cong 0.158 = 15.8\%$$

13

Counting

9.3 Counting Elements of Disjoint Sets: Addition Rule

In this lecture:

❑ Part 1: **Addition Rule**

❑ Part 2: **Difference Rule**

❑ Part 3: **Inclusion Rule**

Apply these rules to count elements of union and disjoint sets

14

The Inclusion/Exclusion Rule

➢ Until now, we learned to count union of sets that they are disjoint.

➢ Now, we learn how to count elements in a union of sets when some of the sets **overlap** (i.e., they are not disjoint)

A B

$A \cap B$

$A \cup B$
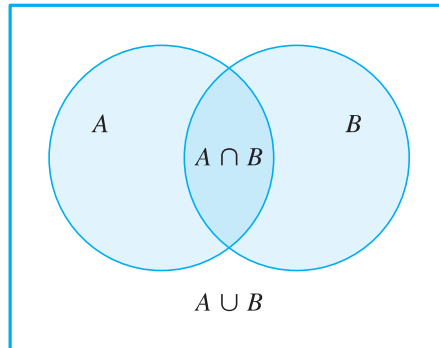
15

Exercise

➢ How many integers from 1 through 1,000 are multiples of 3 or multiples of 5?

16

Exercise

➤ How many integers from 1 through 1,000 are multiples of 3 or multiples of 5?

3s

1	2	3	4	5	6	...	996	997	998	999
		↕			↕		↕			↕
		$3 \cdot 1$			$3 \cdot 2$		$3 \cdot 332$			$3 \cdot 333$

5s

1	2	3	4	5	6	7	8	9	10	...	995	996	997	998	999	1,000
				↕					↕		↕					↕
				$5 \cdot 1$					$5 \cdot 2$		$5 \cdot 199$					$5 \cdot 200$

Overlap

1	2	...	15	...	30	...	975	...	990	...	999	1,000
			↕		↕		↕		↕			
			$15 \cdot 1$		$15 \cdot 2$		$15 \cdot 65$		$15 \cdot 66$			

$$N \, (A \cup B) = N(A) + N(B) - N(A \cap B)$$
$$= 333 \ + 200 \ - \ 66 \qquad = 467$$

17

Exercise

➤ How many integers from 1 through 1,000 are neither multiples of 3 nor multiples of 5?

18

9

Exercise

➢ How many integers from 1 through 1,000 are neither multiples of 3 nor multiples of 5?

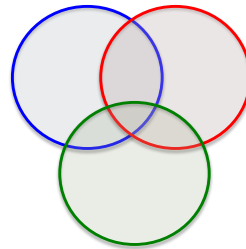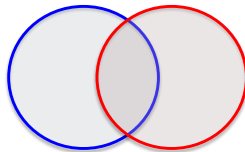$$1{,}000 - 467 = 533$$

The Inclusion/Exclusion Rule

Theorem 9.3.3 The Inclusion/Exclusion Rule for Two or Three Sets

If A, B, and C are any finite sets, then

$$N(A \cup B) = N(A) + N(B) - N(A \cap B)$$

and

$$N(A \cup B \cup C) = N(A) + N(B) + N(C) - N(A \cap B) - N(A \cap C) - N(B \cap C) + N(A \cap B \cap C).$$

Exercise

Given 50 students:

30 took precalculus;
18 took calculus;
26 took Java;
9 took precalculus & calculus;

16 took precalculus & Java;
8 took calculus & Java;
47 took at least 1of the 3 courses.

➤ How many students did not take any of the three courses?

21

Exercise

Given 50 students:

30 took precalculus;
18 took calculus;
26 took Java;
9 took precalculus & calculus;

16 took precalculus & Java;
8 took calculus & Java;
47 took at least 1of the 3 courses.

➤ How many students did not take any of the three courses?

$$50 - 47 = 3.$$

22

Exercise

Given 50 students:

30 took precalculus;
18 took calculus;
26 took Java;
9 took precalculus & calculus;

16 took precalculus & Java;
8 took calculus & Java;
47 took at least 1of the 3 courses.

➤ How many students took all three courses?

P = the set of students who took precalculus
C = the set of students who took calculus
J = the set of students who took Java.

23

Exercise

Given 50 students:

30 took precalculus;
18 took calculus;
26 took Java;
9 took precalculus & calculus;

16 took precalculus & Java;
8 took calculus & Java;
47 took at least 1of the 3 courses.

➤ How many students took all three courses?

P = the set of students who took precalculus
C = the set of students who took calculus
J = the set of students who took Java.

$N(P \cup C \cup J) =$

$$N(P) + N(C) + N(J) - N(P \cap C) - N(P \cap J) - N(C \cap J) + N(P \cap C \cap J)$$

$47 = \quad 30 + 26 + 18 \quad -9 \quad -16 \quad -8 + N(P \cap C \cap J).$

$N(P \cap C \cap J) = 6.$

24

Exercise

Given 50 students:

30 took precalculus;
18 took calculus;
26 took Java;
9 took precalculus & calculus;

16 took precalculus & Java;
8 took calculus & Java;
47 took at least 1of the 3 courses.

➤ How many students took precalculus and calculus but not Java?

Exercise

Given 50 students:

30 took precalculus;
18 took calculus;
26 took Java;
9 took precalculus & calculus;

16 took precalculus & Java;
8 took calculus & Java;
47 took at least 1of the 3 courses.

➤ How many students took precalculus and calculus but not Java?

$= (N(P \cap C)) - (N(P \cap C \cap J) = ?$
$9 \ - \ 6 \ = \ 3$

Exercise

Given 50 students:
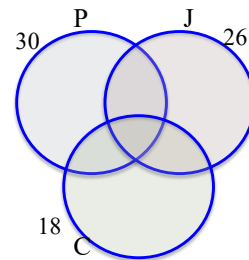
30 took precalculus;
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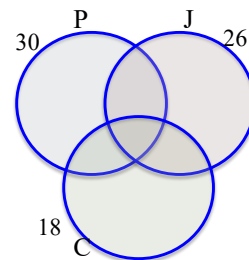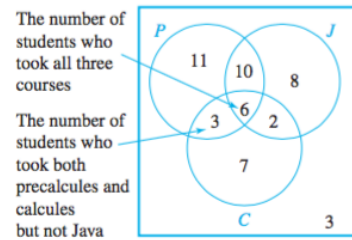8 took calculus & Java;
47 took at least 1of the 3 courses.

➢ How many students took precalculus but neither calculus nor Java?

The number of students who took all three courses

The number of students who took both precalcules and calcules but not Java

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Exercise

Given 50 students:

30 took precalculus;
18 took calculus;
26 took Java;
9 took precalculus & calculus;

16 took precalculus & Java;
8 took calculus & Java;
47 took at least 1of the 3 courses.

➢ How many students took precalculus but neither calculus nor Java?

$N(P) - (N(P∩C)) - N(P∩J)) + N(P∩C∩J) = ?$

$30 - 9 - 16 + 6 = 11$

$N(P) - (N(P∩C)-N(P∩C∩J))-((P∩J) -N(P∩C∩J))$
$30 – 13 – 6 = 11$

The number of students who took all three courses

The number of students who took both precalcules and calcules but not Java