Mustafa Jarrar: Lecture Notes in Discrete Mathematics.
Birzeit University, Palestine, 2015

Counting

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Acknowledgement:
This lecture is based on (but not limited to) to chapter 5 in "Discrete Mathematics with Applications by Susanna S. Epp (3rd Edition)".

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Counting

9.5 Counting Subsets of a Set: Combinations

In this lecture:

- ❑ Part 1: **Permutation versus Combinations**
- ❑ Part 2: **How to Calculate Combinations**
- ❑ Part 3: **Permutations of a Set with Repeated Elements**

Apply these rules to count elements of union and disjoint sets

Counting Subsets of a Set: Combinations (التوافيق)

Suppose 5 members of a group of 12 are to be chosen to work as a team. ***How many distinct five-person teams can be selected?***

كم فريق من 5 اشخاص يمكننا ان نكون من بين 12 شخص؟

Ordering is not important, as the result is a **set**.

- ➢ Recall that we cannot use the r-permutation rule here, because r-permutation produces ordered sets with repetition.

Permutation (التبادیل) Vs. Combinations (التوافیق)

An **ordered** selection of r elements from a set of n elements is **an *r-permutation*** *P(n, r)* of the set.

➔ How many 2-permutations we can produce from {a,b,c,d}

= P(4,2)

An **unordered** selection of r elements from a set of n elements is the same as a subset of size r or an ***r-combination*** of the set.

➔ How many 2-combanations (subsets) can produce from {a,b,c,d}

= $\binom{4}{2}$

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Counting Subsets of a Set: Combinations (التوافیق)

• Definition

Let n and r be nonnegative integers with $r \le n$. An ***r*-combination** of a set of n elements is a subset of r of the n elements. As indicated in Section 5.1, the symbol

$$\binom{n}{r},$$

which is read "n choose r," denotes the number of subsets of size r (r-combinations) that can be chosen from a set of n elements.

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Example 1

Let $S = \{\text{Ann, Bob, Cyd, Dan}\}$. Each committee consisting of three of the four people in S is a 3-combination of S.

List all such 3-combinations of S.

{Bob, Cyd, Dan} leave out Ann
{Ann, Cyd, Dan} leave out Bob
{Ann, Bob, Dan} leave out Cyd
{Ann, Bob, Cyd} leave out Dan.

What is $\binom{4}{3}$?

= 4.

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Example 2

How many unordered selections of 2 elements can be made from the set $\{0, 1, 2, 3\}$?

$\{0, 1\}, \{0, 2\}, \{0, 3\}$ subsets containing 0
$\{1, 2\}, \{1, 3\}$ subsets containing 1 but not already listed
$\{2, 3\}$ subsets containing 2 but not already listed.

Thus $\binom{4}{2} = 6$,

How to calculate $\binom{n}{r}$

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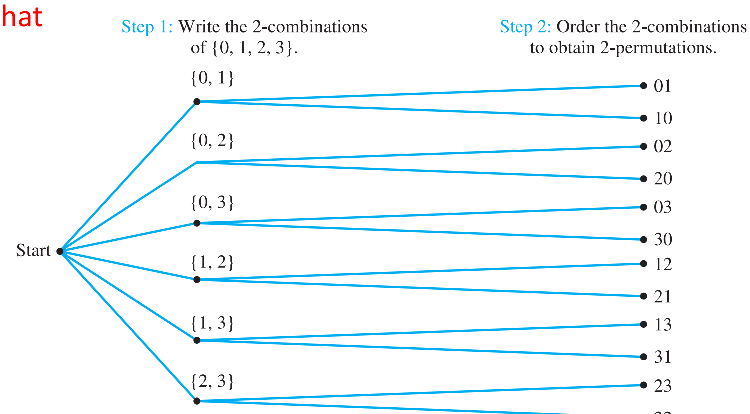
How to calculate $\binom{n}{r}$

What is the relation between Permutations and Combinations?

Notice that



Number of ways in step1 = $\binom{4}{2}$

Number of ways in step2 = **2!**

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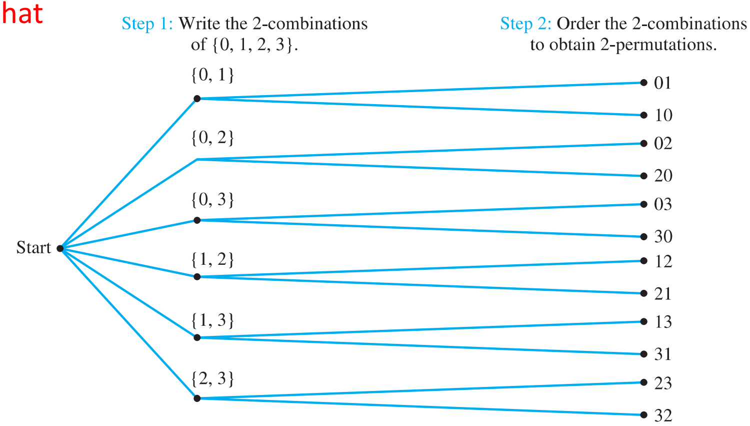
How to calculate $\binom{n}{r}$

What is the relation between Permutations and Combinations?

Notice that



$$P(4, 2) = \binom{4}{2} \cdot 2!$$

,

How to calculate $\binom{n}{r}$

$$P(4, 2) = \binom{4}{2} \cdot 2!.$$

This is an equation that relates $P(4, 2)$ and $\binom{4}{2}$.

$$\binom{4}{2} = \frac{P(4, 2)}{2!}$$

$$\binom{4}{2} = \frac{\frac{4!}{(4-2)!}}{2!} = \frac{4!}{2!(4-2)!} = 6.$$

,

How to calculate $\binom{n}{r}$

$$P(n, r) = \binom{n}{r} \cdot r!.$$

$$\binom{n}{r} = \frac{P(n, r)}{r!}.$$

$$\binom{n}{r} = \frac{\frac{n!}{(n-r)!}}{r!} = \frac{n!}{r!(n-r)!}.$$

Theorem 9.5.1

The number of subsets of size r (or r-combinations) that can be chosen from a set of n elements, $\binom{n}{r}$, is given by the formula

$$\binom{n}{r} = \frac{P(n, r)}{r!} \quad \text{first version}$$

or, equivalently,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \quad \text{second version}$$

where n and r are nonnegative integers with $r \leq n$.

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How to calculate $\binom{n}{0}$

$$\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{1 \cdot n!} = 1$$

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Exercise 1

Suppose 5 members of a group of 12 are to be chosen to work as a team. ***How many distinct five-person teams can be selected?***

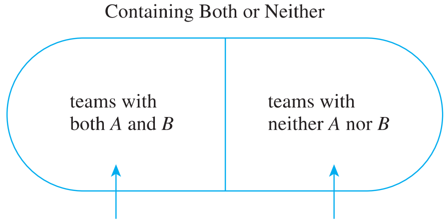
$$\binom{12}{5} = \frac{12!}{5!(12-5)!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7!}{(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) \cdot 7!} = 11 \cdot 9 \cdot 8 = 792.$$

Exercise 2

Suppose 5 members of a group of 12 are to be chosen to work as a team.

Suppose two members of the group of 12 insist on working as a pair - any team must contain either both or neither. How many five-person teams can be formed?

All Possible Five-Person Teams
Containing Both or Neither



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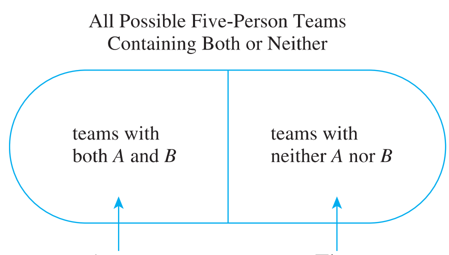

There are $\binom{10}{3} = 120$ of these.

There are $\binom{10}{5} = 252$ of these.

So the total number of teams that contain either both *A* and *B* or neither *A* nor *B* is $120 + 252 = 372$.

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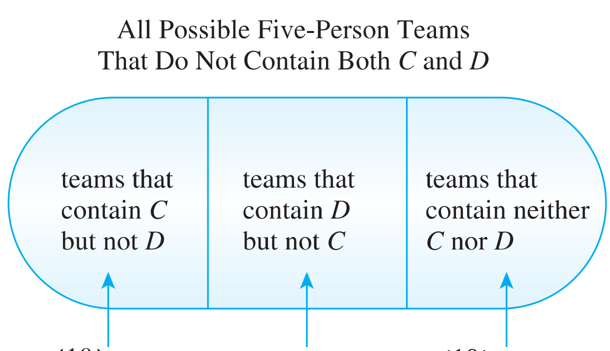
Exercise 3

Suppose 5 members of a group of 12 are to be chosen to work as a team.

Suppose 2 members of the group don't get along and refuse to work together on a team. How many five-person teams can be formed?



$\binom{10}{4} = 210$ $\binom{10}{4} = 210$ $\binom{10}{5} = 252$

So the total number of teams that do not contain both *C* and *D* is $210 + 210 + 252 = 672$.

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Exercise 4

Suppose the group of 12 consists of 5 men and 7 women. How many 5-person teams can be chosen that consist of 3 men and 2 women?

{A, B, C, D, E, **m, n, o, p, q, s, t, r**}

⇩

$\{x_1, x_2, x_3, y_1, y_2\}$

$$\left[\begin{array}{l}\text{number of teams of five that}\\ \text{contain three men and two women}\end{array}\right] = \binom{5}{3}\binom{7}{2} = \frac{5!}{3!2!}\cdot\frac{7!}{2!5!}$$

$$= \frac{7\cdot 6\cdot 5\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1}{\cancel{3}\cdot \cancel{2}\cdot 1\cdot \cancel{2}\cdot 1\cdot \cancel{2}\cdot 1}$$

$$= 210.$$

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Exercise 5

Suppose the group of 12 consists of 5 men and 7 women. How many 5-person teams contain at **least** one man?

$$\left[\begin{array}{l}\text{number of teams}\\ \text{with at least}\\ \text{one man}\end{array}\right] = \left[\begin{array}{l}\text{total number}\\ \text{of teams}\\ \text{of five}\end{array}\right] - \left[\begin{array}{l}\text{number of teams}\\ \text{of five that do not}\\ \text{contain any men}\end{array}\right]$$

$$= \binom{12}{5} - \binom{7}{5} = 792 - \frac{7!}{5!\cdot 2!}$$

$$= 792 - \frac{7\cdot \overset{3}{\cancel{6}}\cdot \cancel{5!}}{\cancel{5!}\cdot \cancel{2}\cdot 1} = 792 - 21 = 771.$$

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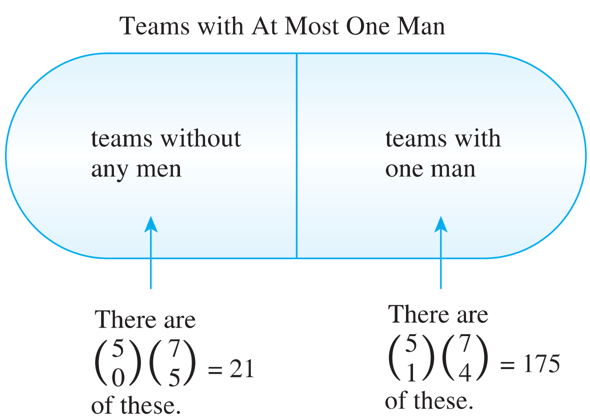
Exercise 6

Suppose the group of 12 consists of 5 men and 7 women.
How many 5-person teams contain at **most** one man?



So the total number of teams with at most one man is 21 + 175 = 196.

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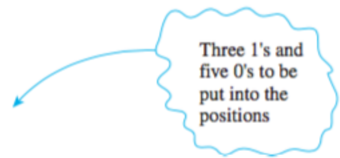
Permutations of a Set with Repeated Elements

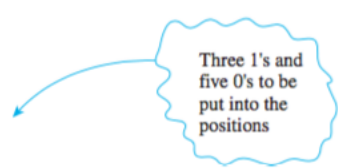
How many eight-bit strings have exactly three 1's?



Permutations of a Set with Repeated Elements

How many eight-bit strings have exactly three 1's?

The number of ways to construct an eight-bit string with exactly three 1's is the same as the number of subsets of three positions that can be chosen from the eight into which to place the 1's.



$$\binom{8}{3} = \frac{8!}{3!\cdot 5!} = \frac{8\cdot 7\cdot 6\cdot 5!}{3\cdot 2\cdot 5!} = 56.$$

Permutations of a Set with Repeated Elements

Consider various ways of ordering the letters in the word *MISSISSIPPI*: *IIMSSPISSIP, ISSSPMIIPIS,* and so on.
How many distinguishable orderings are there?

مثل، لجنة ويحق للشخص تولي اكثر من منصب

__ __ __ __ __ __ __ __ __ __ __
1 2 3 4 5 6 7 8 9 10 11

Step 1: Choose a subset of four positions for the S's.

Step 2: Choose a subset of four positions for the I's.

Step 3: Choose a subset of two positions for the P's.

Step 4: Choose a subset of one position for the M.

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Permutations of a Set with Repeated Elements

Consider various ways of ordering the letters in the word *MISSISSIPPI*: *IIMSSPISSIP, ISSSPMIIPIS,* and so on.
How many distinguishable orderings are there?

$$\begin{bmatrix}\text{number of ways to} \\ \text{position all the letters}\end{bmatrix} = \binom{11}{4}\binom{7}{4}\binom{3}{2}\binom{1}{1}$$

$$= \frac{11!}{4!\,7!}\cdot\frac{7!}{4!\,3!}\cdot\frac{3!}{2!\,1!}\cdot\frac{1!}{1!\,0!}$$

$$= \frac{11!}{4!\cdot 4!\cdot 2!\cdot 1!} = 34{,}650.$$

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Permutations of a Set with Repeated Elements

Theorem 9.5.2 Permutations with sets of Indistinguishable Objects

Suppose a collection consists of n objects of which

n_1 are of type 1 and are indistinguishable from each other

n_2 are of type 2 and are indistinguishable from each other

$\vdots$

n_k are of type k and are indistinguishable from each other,

and suppose that $n_1 + n_2 + \cdots + n_k = n$. Then the number of distinguishable permutations of the n objects is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}$$

$$= \frac{n!}{n_1!\,n_2!\,n_3!\cdots n_k!}.$$

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Double Counting and common mistakes

Read Some tips about counting from the book

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