Form A
Key

Birzeit University
Mathematics Department
First Semester 2022/2023
Math234-Second Exam
Time: 90 minutes
January 28, 2023

Name:.. **Number:**...............

Sections	Instructor Name
(1) and (5)	Dr. Ala Talahmeh
(2)	Dr. Alaeddin Elayyan
(3) and (4)	Dr. Hasan Yousef
(6)	Dr. Mohammad Saleh

15x2 **Exercise#1 [30 marks]**. Answer the following statements as **True** or **False**.

1. (.....F.....) For a matrix A, the row space of A^T is the same as null space of A.
2. (.....F.....) The set of triangular $n \times n$ matrices is a subspace of the vector space $\mathbb{R}^{n\times n}$.
3. (.....T.....) Two subsets of a vector space V that span the same subspace of V need not be equal.
4. (.....F.....) A finite set that contains zero is linearly independent.
5. (.....F.....) If A is a 3×3 matrix with $a_2 = a_3$, then $N(A) = \{0\}$.
6. (.....F.....) For a matrix A, the row space and null space have the same dimensions.
7. (.....F.....) Every set of three vectors in $\mathbb{R}^3$ is a basis for $\mathbb{R}^3$.
8. (.....F.....) If A and B are $n \times n$ matrices that have the same row space, then A and B have the same column space.
9. (.....F.....) If A is a 3×5 matrix, then the rank of A^T is 3.
10. (.....F.....) If U is the REF of a matrix A, then the linearly independent columns of U form a basis for the column space of A.
11. (.....F.....) If A is an $n \times n$ invertible matrix, then the nullity$(A) = n$.
12. (.....F.....) The set $\{1, \cos 2x, \sin^2 x\}$ is linearly independent in $C([0, \pi])$.
13. (.....T.....) The set of all 2×2 matrices with the standard matrix addition and scalar multiplication is a vector space.
14. (.....T.....) There is a set of four vectors that span $\mathbb{R}^4$.
15. (.....F.....) A set with exactly two vectors is linearly dependent if and only if neither vector is a scalar multiple of the other.

15×2

Exercise#2 [30 marks]. Circle the correct answer.

(1) If W is a subspace of a finite-dimensional vector space V, then

- (a) $\dim(W) = \dim(V)$
- (b) $\dim(W) \geq \dim(V)$
- (c) $\dim(W) \leq \dim(V)$
- (d) None of the above

(2) The **dimension** of the subspace $S = \{A \in \mathbb{R}^{2\times 2} : A^T = -A\}$ is equal to

- (a) 1
- (b) 2
- (c) 3
- (d) 0

(3) If A is a 7×6 matrix such that $Ax = 0$ has only the trivial solution, then $\text{rank}(A) =$

- (a) 6
- (b) 0
- (c) 7
- (d) 1

(4) Which of the following is (are) **true**?

- (a) If S is a subspace of a vector space V, then $0 \in S$.
- (b) The set of vectors $\{v, kv\}$ is linearly independent for every scalar k.
- (c) If $f_1, f_2, \ldots\ldots, f_n \in C^{n-1}([a, b])$ and $W(f_1, f_2, \ldots\ldots, f_n)(x) = 0$ for all $x \in [a, b]$, then $f_1, f_2, \ldots\ldots, f_n$ are linearly dependent.
- (d) All are true

(5) Given that $S = \{1, 1 + x + x^2, q(x)\}$ is a basis for P_3. Which of the following is a possible value of $q(x)$?

- (a) 0
- (b) $1 + x$
- (c) -1
- (d) $2 + x + x^2$

(6) The **dimension** of the vector space $C^n([a, b])$ is

(a) $n - 1$

(b) n

(c) infinite

(d) undefined

(7) If A is a 5×6 matrix, then

(a) $\text{rank}(A) \geq 5$

(b) $\text{rank}(A) = 5$

(c) $\text{rank}(A) \leq 5$

(d) $\text{rank}(A) \leq 6$

(8) The **transition matrix** from the ordered basis $\left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ -5 \end{pmatrix} \right\}$ to the standard basis $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ is

(a) $\begin{bmatrix} 1 & -2 \\ 3 & -5 \end{bmatrix}$

(b) $\begin{bmatrix} -5 & 2 \\ -3 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 3 \\ -2 & -5 \end{bmatrix}$

(d) $\begin{bmatrix} -5 & -3 \\ 2 & 1 \end{bmatrix}$

(9) The **Wronskian** of $1, e^x, e^{2x}$ is equal to

(a) $4e^{2x}$

(b) $3e^{2x}$

(c) $2e^{3x}$

(d) $3e^{3x}$

(10) Which of the following does not span $\mathbb{R}^3$?

(a) $\{(2, 2, 2), (0, 0, 3), (0, 1, 1)\}$

(b) $\{(2, -1, 3), (4, 1, 2), (8, -1, 8)\}$

(c) Neither (a) nor (b) span $\mathbb{R}^3$

(d) Both (a) and (b) span $\mathbb{R}^3$

(11) Which of the following is a subspace of $\mathbb{R}^3$?

(a) $S = \{(0, a, a^2)^T : a \in \mathbb{R}\}$
(b) $S = \{(a+2, a, 0)^T : a \in \mathbb{R}\}$
(c) $S = \{(a, b, 2)^T : a, b \in \mathbb{R}\}$
(d) $S = \{(a, b, a-2b)^T : a, b \in \mathbb{R}\}$

(12) Which of the following is a subspace of P_4?

(a) All polynomials $ax^3 + bx^2 + cx + d$ for which a, b, c, d are rational numbers.
(b) All polynomials $ax^3 + bx^2 + cx + d$ for which a, b, c, d are irrational numbers.
(c) All polynomials $ax^3 + bx^2 + cx + d$ for which $a + b + c + d = 0$.
(d) All polynomials $ax^3 + bx^2 + cx + d$ for which $a + b + c + d = 1$.

(13) Which of the following sets in $\mathbb{R}^3$ is linearly dependent?

(a) $\{(1,4,6), (1,-4,0), (4,5,2), (1,3,-5)\}$
(b) $\{(3,2,4), (2,4,3), (0,1,3)\}$
(c) $\{(0,2,0), (2,3,3), (4,2,4)\}$
(d) $\{(1,4,-2), (3,0,0)\}$

(14) Let $S = \{(x+y, x+y, x+2y)^T : x, y \in \mathbb{R}\}$. Then $\dim(S) =$

(a) 1
(b) 2
(c) 3
(d) 0

(15) Let A be a 6×6 matrix with all entries are 1, then nullity$(A) =$

(a) 1
(b) 2
(c) 4
(d) 5

Exercise#3 [15 marks]. Let

$$W = \Big\{p(x) \in P_3 : p''(x) + \int_0^1 p(x)dx = 0\Big\}.$$

(a) Show that W is a **subspace** of P_3.

2 (i) Let $p(x)=0$. Then $p''(x) + \int_0^1 p(x)\,dx = 0 \Rightarrow W \neq \Phi$.

3 (ii) Let $p, q \in W$. Then,

$$(p+q)''(x) + \int_0^1 (p+q)(x)\,dx = \left(p''(x) + \int_0^1 p(x)\,dx\right) + \left(q''(x) + \int_0^1 q(x)\,dx\right)$$
$$= 0 + 0 = 0 \quad (\because p, q \in W).$$

$\therefore p+q \in W$.

3 (iii) Let $\alpha \in \mathbb{R}$ and $p \in W$. Then,

$$(\alpha p)''(x) + \int_0^1 (\alpha p)(x)\,dx = \alpha\left(p''(x) + \int_0^1 p(x)\,dx\right) = \alpha \cdot 0 = 0 \Rightarrow \alpha p \in W.$$

$\square$

(b) Find a **basis** and **dimension** for W.

$p(x) \in P_3 \Rightarrow p(x) = ax^2+bx+c$, $p'(x) = 2ax+b$, $p''(x) = 2a$.

$$p(x) \in W \Rightarrow 2a + \int_0^1 (ax^2+bx+c)\,dx = 0$$
$$\Rightarrow 2a + \left(\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right)\Big|_0^1 = 0$$
$$\Rightarrow 2a + \frac{a}{3} + \frac{b}{2} + c = 0 \Rightarrow \boxed{c = -\frac{7a}{3} - \frac{b}{2}}$$

2

$$\therefore W = \left\{p(x) \in P_3 : p(x) = ax^2 + bx - \frac{7}{3}a - \frac{b}{2}\right\}$$
$$= \left\{p(x) \in P_3 : p(x) = a\left(x^2 - \frac{7}{3}\right) + b\left(x - \frac{1}{2}\right)\right\}$$

1

$= \text{Span}\left\{x^2 - \frac{7}{3}, x - \frac{1}{2}\right\}$ and $\left\{x^2 - \frac{7}{3}, x - \frac{1}{2}\right\}$ lin. indep.

2 2

$\therefore \left\{x^2 - \frac{7}{3}, x - \frac{1}{2}\right\}$ is a basis for W and $\dim W = 2$.

<u>**Exercise#4 [12 marks]**</u>. Let $E = [2x-1, 2x+1]$ and $F = [2+x, 1-x]$ be two ordered bases for P_2.

(a) Find the **transition matrix** from E to the standard basis $\{1, x\}$.

3

$$\begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix}$$

(b) Find the **transition matrix** from the standard basis $\{1, x\}$ to F.

3

$$\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = -\frac{1}{3}\begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{bmatrix}$$

(c) Find the **transition matrix** from E to F.

3

$$S_{E\to F} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}^{-1}\begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix}$$

$$= -\frac{1}{3}\begin{bmatrix} -1 & -3 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1/3 & 1 \\ -5/3 & -1 \end{bmatrix}$$

(d) If $[p(x)]_E = (2, 1)^T$, **use part (c)** to find $[p(x)]_F$.

3

$$[p(x)]_F = S_{E\to F}\,[p(x)]_E$$

$$= \begin{bmatrix} \frac{1}{3} & 1 \\ -\frac{5}{3} & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/3 \\ -13/3 \end{bmatrix}$$

Exercise#5 [18 marks]. Let $A = \begin{bmatrix} 1 & 2 & 2 & 3 & 1 \\ 2 & 4 & 5 & 5 & 4 \\ 3 & 6 & 7 & 8 & 4 \end{bmatrix}$.

(a) Find the **row echelon form** of A.

5

$$\begin{bmatrix} 1 & 2 & 2 & 3 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 3 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & 2 & 3 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Find a basis for the **row space** of A.

2 $\{(1,2,2,3,1), (0,0,1,-1,2), (0,0,0,0,1)\}$.

(c) Find a basis for the **column space** of A.

2 $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix} \right\}$

(d) Find **rank** (A) and **nullity** (A).

2 $\text{rank}(A) = 3$, $\text{nullity}(A) = 5 - \text{rank}(A) = 5 - 3 = 2$.

(e) Find a basis for the **null space** of A.

x_1, x_3, x_5 leading variables. $\boxed{x_2 = \alpha}$, $\boxed{x_4 = \beta}$ free

5

$\boxed{x_5 = 0}$, $x_3 = x_4 - 2x_5 = \beta$

$x_1 = -2x_2 - 2x_3 - 3x_4 - x_5$

$= -2\alpha - 2\beta - 3\beta = -2\alpha - 5\beta$.

$\therefore N(A) = \{(-2\alpha - 5\beta, \alpha, \beta, \beta, 0)^T : \alpha, \beta \in \mathbb{R}\}$

$= \{\alpha(-2,1,0,0,0)^T + \beta(-5,0,1,1,0)^T\}$

$= \text{span}\{\underbrace{(-2,1,0,0,0)^T, (-5,0,1,1,0)^T}_{\text{lin indep}}\}$ 2 are a basis for $N(A)$.

Form B
Key

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Sections	Instructor Name
(1) and (5)	Dr. Ala Talahmeh
(2)	Dr. Alaeddin Elayyan
(3) and (4)	Dr. Hasan Yousef
(6)	Dr. Mohammad Saleh

15x2

Exercise#1 [30 marks]. Answer the following statements as **True** or **False.**

1. (.....T.....) For a matrix A, the row space of A^T is the same as column space of A.
2. (.....T.....) The set of upper triangular $n \times n$ matrices is a subspace of the vector space $\mathbb{R}^{n\times n}$.
3. (.....F.....) Two subsets of a vector space V that span the same subspace of V must be equal.
4. (.....T.....) A finite set that contains zero is linearly dependent.
5. (.....T.....) If A is a 3×3 matrix with $a_2 = a_3$, then $N(A) \neq \{0\}$.
6. (.....T.....) For a matrix A, the row space and column space have the same dimensions.
7. (.....T.....) Every linearly independent set of three vectors in $\mathbb{R}^3$ is a basis for $\mathbb{R}^3$.
8. (.....F.....) If A and B are $n \times n$ matrices that have the same row space, then A and B have the same column space.
9. (.....T.....) If A is a 3×5 matrix, then the rank of A^T is at most 3.
10. (.....F.....) If U is the REF of a matrix A, then the linearly independent columns of U form a basis for the column space of A.
11. (.....T.....) If A is an $n \times n$ invertible matrix, then the rank$(A) = n$.
12. (.....T.....) The set $\{1, \cos 2x, \sin^2 x\}$ is linearly dependent in $C([0, \pi])$.
13. (.....F.....) The set of all 2×2 invertible matrices with the standard matrix addition and scalar multiplication is a vector space.
14. (.....F.....) There is a set of four vectors that span $\mathbb{R}^5$.
15. (.....T.....) A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.

15x2

Exercise#2 [30 marks]. Circle the correct answer.

(1) If V is a subspace of a finite-dimensional vector space W, then

(a) $\dim(W) = \dim(V)$
(b) $\dim(W) \geq \dim(V)$
(c) $\dim(W) \leq \dim(V)$
(d) None of the above

(2) The **dimension** of the subspace $S = \{A \in \mathbb{R}^{2\times 2} : A^T = A\}$ is equal to

(a) 1
(b) 2
(c) 3
(d) 0

(3) If A is a 6×7 matrix with $\operatorname{rank}(A) = 6$, then $\operatorname{nullity}(A) =$

(a) 6
(b) 0
(c) 7
(d) 1

(4) Which of the following is (are) **true**?

(a) If S is a subspace of a vector space V, then $0 \in S$.
(b) The set of vectors $\{v, kv\}$ is linearly dependent for every scalar k.
(c) If $f_1, f_2, \ldots\ldots, f_n \in C^{n-1}([a,b])$ and $f_1, f_2, \ldots\ldots, f_n$ are linearly dependent, then $W(f_1, f_2, \ldots\ldots, f_n)(x) = 0$ for all $x \in [a,b]$.
(d) All are true

(5) Given that $S = \{1, 1+x, q(x)\}$ is a basis for P_3. Which of the following is a possible value of $q(x)$?

(a) 0
(b) $1+x$
(c) -1
(d) $2+x+x^2$

(6) The **dimension** of the vector space P_n is

(a) $n - 1$

(b) n

(c) infinite

(d) undefined

(7) If A is a 3×5 matrix, then

(a) $\text{rank}(A) \geq 3$

(b) $\text{rank}(A) \leq 3$

(c) $\text{rank}(A) = 3$

(d) $\text{rank}(A) = 5$

(8) The **transition matrix** from the standard basis $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ to the ordered basis $\left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ -5 \end{pmatrix} \right\}$ to is

(a) $\begin{bmatrix} 1 & -2 \\ 3 & -5 \end{bmatrix}$

(b) $\begin{bmatrix} -5 & 2 \\ -3 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 3 \\ -2 & -5 \end{bmatrix}$

(d) $\begin{bmatrix} -5 & -3 \\ 2 & 1 \end{bmatrix}$

(9) The **Wronskian** of $1, x, e^{2x}$ is equal to

(a) $4e^{2x}$

(b) $3e^{2x}$

(c) $2e^{3x}$

(d) $3e^{3x}$

(10) Which of the following does not span $\mathbb{R}^3$?

(a) $\{(2, 2, 2), (0, 0, 3), (0, 0, 1)\}$

(b) $\{(2, -1, 3), (4, 1, 2), (8, -1, 8)\}$

(c) Neither (a) nor (b) span $\mathbb{R}^3$

(d) Both (a) and (b) span $\mathbb{R}^3$

(11) Which of the following is a subspace of $\mathbb{R}^3$?

(a) $S = \{(0, a, a^2)^T : a \in \mathbb{R}\}$

(b) $S = \{(a+2, a, 0)^T : a \in \mathbb{R}\}$

(c) $S = \{(a, b, 0)^T : a, b \in \mathbb{R}\}$

(d) $S = \{(a, b, ab)^T : a, b \in \mathbb{R}\}$

(12) Which of the following is a subspace of P_4?

(a) All polynomials $ax^3 + bx^2 + cx + d$ for which a, b, c, d are rational numbers.

(b) All polynomials $ax^3 + bx^2 + cx + d$ for which a, b, c, d are real numbers.

(c) All polynomials $x^3 + bx^2 + cx + d$ for which $b + c + d = 0$.

(d) All polynomials $ax^3 + bx^2 + cx + d$ for which $a + b + c + d = 1$.

(13) Which of the following sets in $\mathbb{R}^3$ is linearly dependent?

(a) $\{(1, 4, 6), (1, -4, 0), (4, 5, 2)\}$

(b) $\{(3, 2, 4), (2, 4, 3), (0, 1, 3), (1, -2, 1)\}$

(c) $\{(0, 2, 0), (2, 3, 3), (4, 2, 4)\}$

(d) $\{(1, 4, -2), (3, 0, 0)\}$

(14) Let $S = \{(x+y, x+y, x+y)^T : x, y \in \mathbb{R}\}$. Then $\dim(S) =$

(a) 1

(b) 2

(c) 3

(d) 0

(15) Let A be a 6×6 matrix with all entries are 1, then $\text{rank}(A) =$

(a) 1

(b) 2

(c) 4

(d) 5

Exercise#3 [15 marks]. Let

$$W = \Big\{p(x) \in P_3 : p'(1) + \int_0^1 p(x)dx = 0\Big\}.$$

(a) Show that W is a **subspace** of P_3.

2 (i) Let $p(x) = 0$. Then $p'(1) + \int_0^1 p(x)\,dx = 0 \Rightarrow W \neq \phi$.

3 (ii) Let $p, q \in W$. Then,

$$(p+q)'(1) + \int_0^1 (p+q)(x)\,dx = \left(p'(1) + \int_0^1 p(x)\,dx\right) + \left(q'(1) + \int_0^1 q(x)\,dx\right)$$

$$= 0 + 0 = 0 \; (\because p, q \in W$$

$\therefore p+q \in W$.

3 (iii) Let $\alpha \in \mathbb{R}$ and $p \in W$. Then,

$$(\alpha p)'(1) + \int_0^1 (\alpha p)(x)\,dx = \alpha\left(p'(1) + \int_0^1 p(x)\,dx\right) = \alpha \cdot 0 = 0$$

$\Rightarrow \alpha p \in W$. $\square$

(b) Find a **basis** and **dimension** for W.

$p(x) \in P_3 \Rightarrow p(x) = ax^2 + bx + c$, $p'(x) = 2ax + b$, $p'(1) = 2a + b$.

$$p(x) \in W \Rightarrow 2a + b + \int_0^1 (ax^2 + bx + c)\,dx = 0$$

$$2a + b + \left(\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right)\Big|_0^1 = 0$$

$$\Rightarrow 2a + b + \frac{a}{3} + \frac{b}{2} + c = 0 \Rightarrow \boxed{c = -\frac{7a}{3} - \frac{3b}{2}}$$

2 $\therefore W = \left\{p(x) \in P_3 : p(x) = ax^2 + bx - \frac{7}{3}a - \frac{3b}{2}\right\}$

$= \left\{p(x) \in P_3 : p(x) = a\left(x^2 - \frac{7}{3}\right) + b\left(x - \frac{3}{2}\right)\right\}$

$= \text{span}\left\{x^2 - \frac{7}{3}, x - \frac{3}{2}\right\}$ and $\left\{x^2 - \frac{7}{3}, x - \frac{3}{2}\right\}$ lin. indep. 1

$\therefore \left\{x^2 - \frac{7}{3}, x - \frac{3}{2}\right\}$ is 2 a basis for W and $\dim W = 2$

2

Exercise#4 [12 marks]. Let $E = [2+x, 1-x]$ and $F = [2x-1, 2x+1]$ be two ordered bases for P_2.

(a) Find the **transition matrix** from E to the standard basis $\{1, x\}$.

3

$$\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

(b) Find the **transition matrix** from the standard basis $\{1, x\}$ to F.

3

$$\begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix}^{-1} = -\frac{1}{4}\begin{bmatrix} 2 & -1 \\ -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 & 1/4 \\ 1/2 & 1/4 \end{bmatrix}$$

(c) Find the **transition matrix** from E to F.

3

$$S_{E\to F} = \begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} 2 & -1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= -\frac{1}{4}\begin{bmatrix} 3 & 3 \\ -5 & -1 \end{bmatrix} = \begin{bmatrix} -3/4 & -3/4 \\ 5/4 & 1/4 \end{bmatrix}$$

(d) If $[p(x)]_E = (1,2)^T$, **use part (c)** to find $[p(x)]_F$.

3

$$[p(x)]_F = S_{E\to F}\,[p(x)]_E$$

$$= -\frac{1}{4}\begin{bmatrix} 3 & 3 \\ -5 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= -\frac{1}{4}\begin{bmatrix} 9 \\ -7 \end{bmatrix} = \begin{bmatrix} -9/4 \\ 7/4 \end{bmatrix}$$

Exercise#5 [18 marks]. Let $A = \begin{bmatrix} 1 & 3 & 3 & 2 & 1 \\ 2 & 6 & 7 & 3 & 2 \\ 3 & 9 & 10 & 5 & 4 \end{bmatrix}$.

(a) Find the **row echelon form** of A.

5

$$\begin{bmatrix} 1 & 3 & 3 & 2 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 3 & 2 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Find a basis for the **row space** of A.

2 $\{(1,3,3,2,1), (0,0,1,-1,0), (0,0,0,0,1)\}$

(c) Find a basis for the **column space** of A.

2 $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \\ 10 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \right\}$

(d) Find **rank** (A) and **nullity** (A).

2 rank$(A) = 3$, nullity$(A) = 5 - \text{rank}(A) = 2$.

(e) Find a basis for the **null space** of A.

x_1, x_3, x_5 leading variables. $x_2 = \alpha$, $x_4 = \beta$ free

$x_5 = 0$, $x_3 = x_4 = \beta$

5

$x_1 = -3x_2 - 3x_3 - 2x_4 - x_5 = -3\alpha - 5\beta$.

$= -3\alpha - 3\beta - 2\beta$

$N(A) = \{(-3\alpha - 5\beta, \alpha, \beta, \beta, 0)^T : \alpha, \beta \in \mathbb{R}\}$

$= \{\alpha(-3,1,0,0,0)^T + \beta(-5,0,1,1,0)^T : \alpha, \beta \in \mathbb{R}\}$

$= \text{span}\{(-3,1,0,0,0)^T, (-5,0,1,1,0)^T\}$ 2

lin indep. and form a basis for $N(A)$.