جامعة بيرزيت

BIRZEIT UNIVERSITY

MATH DEPARTMENT
MATH243, Second Exam

1^{st} Semester 2022- 2023

Student Name :____________ **ID :** _________

1) (8 points) Use induction to prove the followings

a) For every $n \in N, 6\ divides\ n^3 - n$

using induction,

(1) (1) To show its true for $n=1$, $1-1=0$
since $\frac{0}{6} = 0 \in \mathbb{Z}$, so its true for $n=1$

(1) (2) Assume its true for $n=k$, i.e. $k^3-k=6m$, $m \in \mathbb{Z}$

(3) To show its true for $n=k+1$, i.e to show that
$(k+1)^3-(k+1)=6j, j\in \mathbb{Z}$.

But $(k+1)^3-(k+1)=(k+1)[(k+1)^2-1]=(k+1)(k^2+2k+1-1)$

(2pts) $=(k+1)(k^2+2k) = k(k+1)(k+2) = k(k^2+3k+2)$
$= k^3+3k^2+2k$

b) For every $n \in N, n \geq 2,\qquad 2^{n+1} < 3^n$

$= \underbrace{k^3-k}+3k^2+3k$

$* = 6m + 3k^2+3k$
$+ 3(\underbrace{k^2+k}_{2z,\ z\in \mathbb{Z}})$

$k^2+k = 2z$
$\parallel$
$k(k+1)$
if k is odd, $k+1$ even $\Rightarrow k(k+1)$ is even
if k is even $k+1$ odd $\Rightarrow k(k+1)$ is even

so $* = 6(\underbrace{m+z}_{\in \mathbb{Z}})$.

(b) To show $2^{n+1} < 3^n$, $\forall n \geq 2$

(1) (1) To show its true for $n=2$

which is true since $2^3 = 8$
is less than $3^2 = 9$

(1) (2) Assume its true for $n=k$

i.e $2^{k+1} < 3^k$

(3) To show its true for $n=k+1$

i.e $2^{k+2} < 3^{k+1}$

(2) But $2^{k+2} = 2\,2^{k+1} < 2 \cdot 3^k$ induction step (2)

$< 3 \cdot 3^k = 3^{k+1}$

2) (6 points) Let $S = R \times R$, and for any two members $(a,b) \cong (c,d)$ of S if $ad = bc$.

a) Prove that $\cong$ is an equivalence relation

(1) Reflexive, $\forall (a,b) \in \mathbb{R}\times\mathbb{R}$

$(a,b) \simeq (a,b)$ since $ab = ab$

(2) Symmetric,

suppose $(a,b) \sim (c,d)$, to show $(c,d) \sim (a,b)$

since $(a,b) \sim (c,d) \Rightarrow ad = bc$

$\Rightarrow bc = ad \Rightarrow (c,d) \sim (a,b)$

(3) transitive

If $(a,b) \sim (c,d)$ & $(c,d) \sim (e,f)$, To show $(a,b) \sim (e,f)$

i.e $ad = bc$ & $cf = ed$ i.e $af = eb$

$$\Rightarrow af = \underset{c \neq 0}{a}\left(\frac{ed}{c}\right) = \frac{e\,ad}{c} = \frac{e(bc)}{c} = eb$$

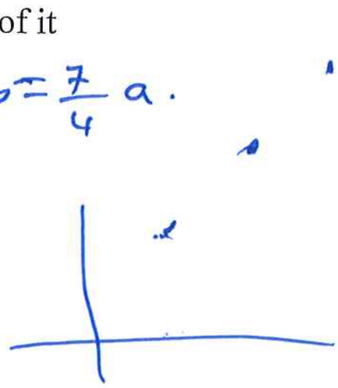
b) Describe geometrically [(4,7)] and list three members of it

$(4,7) \sim (a,b) \Rightarrow 4b = 7a. \Rightarrow b = \frac{7}{4}a.$

$[(4,7)] = \{(4,7), (8,14), (12,21), --$

Discrete line with slope $\frac{7}{4}$

and pass through $(4,7)$



3) (6 points) a) Let $S=\{(x,y) \in R \times R, y = x^3 + x\}$

$$H = \{(x,y) \in R \times R, y^3 = x - 1\}$$

Find $S°H$, $H°H$

$S \circ H = \{(x,y) : \exists z \text{ s.t } (x,z) \in H \text{ and } (z,y) \in S\}$

$z^3 = x-1$ & $y = z^3 + z$

$\Rightarrow y = (x-1) + \sqrt[3]{x-1}$

3

$H \circ S = \{(x,y) : \exists z \text{ s.t } (x,z) \in S \text{ \& } (z,y) \in H\}$

$z = x^3 + x$ & $y^3 = z - 1$

& $y^3 = x^3 + x - 1$

$y = \sqrt[3]{x^3 + x - 1}$

b) (6 points) Let R be a relation on a set A, Show that R is transitive if and only if $R°R \subseteq R$

$H \circ H = \{(x,y) : \exists z \text{ s.t } (x,z) \in H \text{ \& } (z,y) \in H\}$

$z^3 = x-1$ & $y^3 = z-1$

$y^3 = \sqrt[3]{x-1} - 1$

$y = \sqrt[3]{\sqrt[3]{x-1} - 1}$

3

(b) Let R be a relation on A, To show

R is transitive $\Leftrightarrow R \circ R \subseteq R$

($\Rightarrow$) Let $(x,y) \in R \circ R$

$\Rightarrow \exists z$ s.t. $(x,z) \in R$ & $(z,y) \in R$

But R is transitive $\Rightarrow (x,y) \in R$

3

$\Rightarrow R \circ R \subseteq R$

($\Leftarrow$) Let $R \circ R \subseteq R$, To show R is transitive

Let $(x,y) \in R$ & $(y,z) \in R$

then $(x,z) \in R \circ R \subseteq R$

3

$\Rightarrow (x,z) \in R$

$\Rightarrow R$ is transitive

4) (6 points) Let $A_b = \{(x, y) \in R \times R, y = 2x + b\}$

And let $H = \{A_b, b \in R\}$ Show that H is a partition on $R \times R$

1 ① $H \neq \phi$, $A_0 \in H$

② suppose $A_b \cap A_a \neq \phi$, $\exists (x,y) \in A_b \cap A_a$

2 $\Rightarrow y = 2x + b = 2x + a$

$\Rightarrow b = a \Rightarrow A_b = A_a$.

③ $\cup A_b = \mathbb{R} \times \mathbb{R}$

1.5 ($\subseteq$) let $(x,y) \in \cup A_b \Rightarrow (x,y) \in A_b$ for some $b \Rightarrow (x,y) \in \mathbb{R} \times \mathbb{R}$

~~$\Rightarrow y = 2x+b$~~

1.5 ($\supseteq$) let $(x,y) \in \mathbb{R} \times \mathbb{R} \Rightarrow y - 2x = b$ for some b.

$\Rightarrow y = 2x + b \Rightarrow y \in A_b$

5) (6 points) Let R be an equivalence relation on a nonempty set S and let H be the family of all equivalence classes on S, Show that H is a partition on S.

1 ① H is not empty, since S is not empty, $\exists x \in S$, and $x \in [x]$ & $[x] \in H$

1 ② $\forall x \in S$, $x \in [x]$ $\Rightarrow$ so $\bigcup_{x \in S} [x] = S$

③ If $[x] \cap [y] \neq \phi$, to show $[x] = [y]$

4 let $z \in [x] \cap [y]$, $z \in [x]$ & $z \in [y]$ so $(x,z) \in R$ & $(y,z) \in R$ and $(z,x) \in R$ & $(z,y) \in R$ (since R is E. R)

let $w \in [x] \Rightarrow (x,w) \in R$

$\Rightarrow (w,x) \in R$ and since $(x,z) \in R$

$\Rightarrow (w,z) \in R$ (R is transitive)

so $(w,z) \in R$ & $(z,y) \in R \Rightarrow (w,y) \in R$

$\Rightarrow (y,w) \in R \Rightarrow w \in [y]$

5)(12points) Prove or disprove

a) $(S \cup R)[x] = S[x] \cup R[x]$ True

(⇒) let $y \in (S \cup R)[x] \Rightarrow (x,y) \in S$ or $(x,y) \in R$

$\Rightarrow y \in S[x]$ or $y \in R[x]$

$\Rightarrow y \in S[x] \cup R[x]$

(⇐) If $y \in S[x]$ or $y \in R[x]$

$\Rightarrow (x,y) \in S$ or $(x,y) \in R$

$\Rightarrow (x,y) \in S \cup R \Rightarrow y \in (S \cup R)[x]$

b)The relation $S = \{(x,y) \in R \times R, |x - y| = 2\}$is transitive.

$(2,0) \in S \Rightarrow |2-0| = 2$

& $(0,2) \in S \Rightarrow |0-2| = 2$

But $(2,2) \notin S \Rightarrow |2-2| = 0$

d)If the relations R and S are reflexive on A then R°S is also reflexive .

$\forall x, (x,x) \in R$, and $(x,x) \in S$

$\Rightarrow (x,x) \in R \circ S$

e) If the relations R and S are symmetric then R°S is also symmetric .

$R = \{(2,3),(3,2)\}$

$S = \{(2,2)\}$

$R \circ S = \{(2,3)\}$ Not sym