



BIRZEIT UNIVERSITY

MATH DEPARTMENT  
MATH243, Second Exam

1<sup>st</sup> Semester 2022-2023

Student Name : \_\_\_\_\_

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1) (8 points) Use induction to prove the followings

a) For every  $n \in \mathbb{N}$ , 6 divides  $n^3 - n$

using induction,

① ① To show its true for  $n=1$ ,  $1-1=0$   
Since  $\frac{0}{6} = 0 \in \mathbb{Z}$ , so its true for  $n=1$

① ② Assume its true for  $n=k$ , i.e.  $k^3 - k = 6m$ ,  $m \in \mathbb{Z}$

③ To show its true for  $n=k+1$ , i.e. to show that  
 $(k+1)^3 - (k+1) = 6j$ ,  $j \in \mathbb{Z}$ .

But

② MB

$$\begin{aligned} (k+1)^3 - (k+1) &= (k+1)[(k+1)^2 - 1] = (k+1)(k^2 + 2k + 1 - 1) \\ &= (k+1)(k^2 + 2k) = k(k+1)(k+2) = k(k^2 + 3k + 2) \\ &= k^3 + 3k^2 + 2k \end{aligned}$$

b) For every  $n \in \mathbb{N}$ ,  $n \geq 2$ ,  $2^{n+1} < 3^n$

$$= k^3 + 3k^2 + 3k$$

$$\begin{aligned} * &= 6m + 3k^2 + 3k \\ &\quad + 3(k^2 + k) \\ &\quad \quad \quad \underbrace{\qquad\qquad\qquad}_{2\mathbb{Z}, 2 \in \mathbb{Z}} \end{aligned}$$

$$k^2 + k = 2z$$

$$\parallel$$
$$k(k+1)$$

if  $k$  is odd,  $k+1$  even  $\Rightarrow k(k+1)$  is even  
if  $k$  is even,  $k+1$  odd  $\Rightarrow k(k+1)$  is even

$$\text{so } * = 6(m+2) \in \mathbb{Z}$$

(b) To show  $2^{n+1} < 3^n$ ,  $\forall n \geq 2$

(1) (1) To show it's true for  $n=2$

which is true since  $2^3=8$   
is less than  $3^2=9$

(1) (2) Assume it's true for  $n=k$

i.e.  $2^{k+1} < 3^k$

(3) To show it's true for  $n=k+1$

i.e.  $2^{k+2} < 3^{k+1}$

(2) But  $2^{k+2} = 2 \cdot 2^{k+1} < 2 \cdot 3^k$  inductive  
 $< 3 \cdot 3^k = 3^{k+1}$  step (2)

2) (6 points) Let  $S = \mathbb{R} \times \mathbb{R}$ , and for any two members  $(a, b) \cong (c, d)$  of  $S$  if  $ad = bc$ .

a) Prove that  $\cong$  is an equivalence relation

(1) Reflexive,  $\forall (a, b) \in \mathbb{R} \times \mathbb{R}$   
 $(a, b) \cong (a, b)$  since  $ab = ab$

(2) Symmetric,  
 Suppose  $(a, b) \cong (c, d)$ , to show  $(c, d) \cong (a, b)$   
 since  $(a, b) \cong (c, d) \Rightarrow ad = bc$   
 $\Rightarrow bc = ad \Rightarrow (c, d) \cong (a, b)$

(3) transitive

$\Rightarrow (a, b) \cong (c, d) \wedge (c, d) \cong (e, f)$ , To show  $(a, b) \cong (e, f)$   
 i.e.  $ad = bc \wedge ct = ed$  i.e.  $af = eb$

$$\Rightarrow af = a \left( \frac{ed}{c} \right) = \frac{ead}{c} = \frac{e(bc)}{c} = eb$$

b) Describe geometrically  $[(4, 7)]$  and list three members of it

$$(4, 7) \cong (a, b) \Rightarrow 4b = 7a. \Rightarrow b = \frac{7}{4}a.$$

$$[(4, 7)] = \left\{ (4, 7), (8, 14), (12, 21), \dots \right\}$$

Discrete line with slope  $\frac{7}{4}$   
 and passing through  $(4, 7)$





3) (6 points) a) Let  $S = \{(x, y) \in \mathbb{R} \times \mathbb{R}, y = x^3 + x\}$

$$H = \{(x, y) \in \mathbb{R} \times \mathbb{R}, y^3 = x - 1\}$$

Find  $S \circ H, H \circ S$

$$S \circ H = \{(x, y) : \exists z \text{ s.t. } (x, z) \in H \text{ and } (z, y) \in S\}$$

$$z^3 = x - 1 \quad \wedge \quad y = z^3 + z$$

$$\Rightarrow y = (x - 1) + \sqrt[3]{x - 1}$$

$$H \circ S = \{(x, y) : \exists z \text{ s.t. } (x, z) \in S \wedge (z, y) \in H\}$$

$$z = x^3 + x \quad \wedge \quad y^3 = z - 1$$

$$\wedge y^3 = x^3 + x - 1$$

$$y = \sqrt[3]{x^3 + x - 1}$$

b) (6 points) Let  $R$  be a relation on a set  $A$ , Show that  $R$  is transitive if and only if  $R \circ R \subseteq R$

$$H \circ H = \{(x, y) : \exists z \text{ s.t. } (x, z) \in H \wedge (z, y) \in H\}$$

$$z^3 = x - 1 \quad \wedge \quad y^3 = z - 1$$

$$y^3 = \sqrt[3]{x - 1} - 1$$

$$y = \sqrt[3]{\sqrt[3]{x - 1} - 1}$$

(b) Let  $R$  be a relation on  $A$ , To show  
 $R$  is transitive  $\Leftrightarrow R \circ R \subseteq R$

$\Rightarrow$  Let  $(x, y) \in R \circ R$

$\Rightarrow \exists z$  st  $(x, z) \in R$  &  $(z, y) \in R$

But  $R$  is transitive  $\Rightarrow (x, y) \in R$

3

$\Rightarrow R \circ R \subseteq R$

( $\Leftarrow$ ) Let  $R \circ R \subseteq R$ , To show  $R$  is transitive

Let  $(x, y) \in R$  &  $(y, z) \in R$

then  $(x, z) \in R \circ R \subseteq R$

3

$\Rightarrow (x, z) \in R$

$\Rightarrow R$  is transitive

4) (6 points) Let  $A_b = \{(x, y) \in \mathbb{R} \times \mathbb{R}, y = 2x + b\}$

And let  $H = \{A_b, b \in \mathbb{R}\}$  Show that  $H$  is a partition on  $\mathbb{R} \times \mathbb{R}$

(1)

(1)  $H \neq \emptyset, A_b \in H$

(2) suppose  $A_b \cap A_a \neq \emptyset, \exists (x, y) \in A_b \cap A_a$

(2)

$$\Rightarrow y = 2x + b = 2x + a$$

$$\Rightarrow b = a \Rightarrow A_b = A_a.$$

(3)  $\cup A_b = \mathbb{R} \times \mathbb{R}$

1.5 (1)  $\forall (x, y) \in \cup A_b \Rightarrow (x, y) \in A_b$  for some  $b \Rightarrow (x, y) \in \mathbb{R} \times \mathbb{R}$   
 ~~$\Rightarrow y = 2x + b$~~

1.5 (2)  $\forall (x, y) \in \mathbb{R} \times \mathbb{R} \Rightarrow y - 2x = b$  for some  $b$ .  
 $\Rightarrow y = 2x + b \Rightarrow (x, y) \in A_b$

5) (6 points) Let  $R$  be an equivalence relation on a nonempty set  $S$  and let  $H$  be the family of all equivalence classes on  $S$ , Show that  $H$  is a partition on  $S$ .

(1) (1)  $H$  is not empty, since  $S$  is not empty,  $\exists x \in S$ , and  $x \in [x]$   
 $\downarrow [x] \in H$

(1) (2)  $\forall x \in S, x \in [x] \Rightarrow \cup_{x \in S} [x] = S$

(3)  $\nexists [x] \cap [y] \neq \emptyset$ , to show  $[x] = [y]$

(4)  $\forall z \in [x] \cap [y], z \in [x] \downarrow z \in [y]$  so  $(x, z) \in R \downarrow (y, z) \in R$   
 and  $(z, x) \in R \downarrow (z, y) \in R$   
 (since  $R$  is  $\subseteq R$ )

$\forall w \in [x] \Rightarrow (x, w) \in R$

$\Rightarrow (w, x) \in R$  and since  $(x, z) \in R$

$\Rightarrow (w, z) \in R$  ( $R$  is transitive)

so  $(w, z) \in R \downarrow (z, y) \in R \Rightarrow (w, y) \in R$

$= (y, w) \in R \Rightarrow w \in [y]$

The first part of the document discusses the importance of maintaining accurate records of all transactions. It emphasizes that every entry should be supported by a valid receipt or invoice. This ensures transparency and allows for easy verification of the data.

Furthermore, it is noted that the records should be kept in a secure and accessible location. Regular audits are recommended to identify any discrepancies or errors early on. This proactive approach helps in maintaining the integrity of the financial information.

In addition, the document highlights the need for clear communication between all parties involved. Regular meetings and reports should be used to keep everyone informed about the current status and any upcoming changes.

The second section focuses on the implementation of a robust internal control system. This involves defining clear roles and responsibilities for each employee. It also includes the establishment of a strong code of ethics and a zero-tolerance policy for any form of fraud or misconduct.

Key components of this system include:

- Segregation of duties to prevent any one individual from having too much control over a process.
- Regular reconciliation of accounts to ensure that all entries are balanced.
- Use of technology to automate repetitive tasks and reduce the risk of human error.

Finally, the document concludes by stating that a strong internal control system is essential for the long-term success and sustainability of any organization. It provides a framework for identifying and mitigating risks, ensuring that the organization remains compliant with all applicable laws and regulations.



5)(12points) Prove or disprove

a)  $(S \cup R)[x] = S[x] \cup R[x]$  True

$\Rightarrow$  Let  $y \in (S \cup R)[x] \Rightarrow (x, y) \in S$  or  $(x, y) \in R$   
 $\Rightarrow y \in S[x]$  or  $y \in R[x]$   
 $\Rightarrow y \in S[x] \cup R[x]$

$\Leftarrow$  If  $y \in S[x]$  or  $y \in R[x]$   
 $\Rightarrow (x, y) \in S$  or  $(x, y) \in R$   
 $\Rightarrow (x, y) \in S \cup R \Rightarrow y \in (S \cup R)[x]$

b) The relation  $S = \{(x, y) \in R \times R, |x - y| = 2\}$  is transitive.

$(2, 0) \in S \Rightarrow |2 - 0| = 2$

$(0, 2) \in S \Rightarrow |0 - 2| = 2$

But  $(2, 2) \notin S \Rightarrow |2 - 2| = 0$

d) If the relations R and S are reflexive <sup>on A</sup> then  $R \circ S$  is also reflexive.

$\forall x, (x, x) \in R$ , and  $(x, x) \in S$

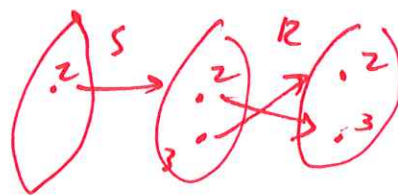
$\Rightarrow (x, x) \in R \circ S$

e) If the relations R and S are symmetric then  $R \circ S$  is also symmetric.

$R = \{(2, 3), (3, 2)\}$

$S = \{(2, 2), (3, 3)\}$

$R \circ S = \{(2, 3)\}$  Not sym





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