جامعة بيرزيت
BIRZEIT UNIVERSITY

Mathematics Department

Math 243

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Exam 1st. Semester 2022/2023

Student name: .... ID no.: ....

points) a) Show that if $f(x)$ is strictly decreasing then $f(x)$ is 1-1 and f^{-1} is strictly decreasing.

ppose f is strictly decreasing, i.e if $a<b$ then $f(a)>f(b)$

to show f is 1-1,

$x_1 \neq x_2$, then either $x_1 < x_2$ or $x_2 < x_1$

if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ since f is strictly decreas-

and so $f(x_1) \neq x_2$

Similarly if $x_2 > x_1$.

show f^{-1} is strictly decreasing, let $c<d$ to show $f^{-1}(c) > f^{-1}(d)$

$$\left.\begin{array}{l} \text{But } f^{-1}(c) = x_1 \Rightarrow f(x_1) = c \\ f^{-1}(d) = x_2 \Rightarrow f(x_2) = d \end{array}\right\} \text{ if } f^{-1}(c) \leq f^{-1}(d) \Rightarrow x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2)$$

$c \geq d$ ✗.

points) Show that if S is nonempty set and $f, g \in sym(S)$ then $f^\circ g \in sym(S)$

Proof: let f & $g \in Sym\,S$, f is 1-1 and onto and so is g.

To show $f \circ g$ is 1-1

Suppose $(f\circ g)(x_1) = (f \circ g)(x_2)$

$\Rightarrow f(g(x_1)) = f(g(x_2))$

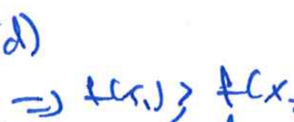
$\Rightarrow g(x_1) = g(x_2)$ since f is 1-1

$\Rightarrow x_1 = x_2$ since g is 1-1

$\Rightarrow f\circ g$ is 1-1

To show $f \circ g$ is onto, let $y \in S$

since f is onto, $\exists\, x \in S$ s.t $f(x) = y$



But since g is onto, $\exists\, z \in S$ s.t $g(z) = x$

and so $f(g(z)) = f(x) = y$.

Let $f: R \to (-1,1)$ be defined by $f(x) = \frac{x}{1+|x|}$ Show that f is 1-1 and onto and

To show f is 1-1

let $f(x_1) = f(x_2)$ to show $x_1 = x_2$

pts $\frac{x_1}{1+|x_1|} = \frac{x_2}{1+|x_2|} \Rightarrow x_1 + x_1|x_2| = x_2 + x_2|x_1|$ 3 pts

4 cases

To show f is onto,

let $y \in (0,1] \Rightarrow y = \frac{x_0}{1+|x_0|} \Rightarrow x_0 > 0$ $\Rightarrow y = \frac{x}{1+x}$

$\Rightarrow y + yx = x \Rightarrow y = x - yx = x(1-y)$

$\Rightarrow x = \frac{y}{1-y}$

$f(x) = f\left(\frac{y}{1-y}\right) = \frac{y/1-y}{1+\left|\frac{y}{1-y}\right|} = \frac{y/1-y}{1+\frac{y}{1-y}} = \frac{y/\cancel{1-y}}{\frac{\cancel{1-y}+y}{\cancel{1-y}}} = y$

if $y \in (-1,0] \Rightarrow y = \frac{x}{1+|x|} \Rightarrow x < 0, |x| = -x$

$\Rightarrow y = \frac{x}{1-x} \Rightarrow y - yx = x \Rightarrow y = yx + x = x(y+1)$

$\Rightarrow x = \frac{y}{1+y}$ so $f(x) = f\left(\frac{y}{1+y}\right) = \frac{\frac{y}{1+y}}{1+\left|\frac{y}{1+y}\right|} = \frac{\frac{y}{1+y}}{1-\frac{y}{1+y}} = y$

from above

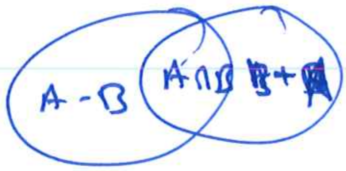
$f^{-1}(x) = \begin{cases} \frac{x}{1-x} & , 1 > x \geq 0 \\ \frac{x}{1+x} & , -1 < x < 0 \end{cases} = \frac{x}{1-|x|}$

points)Show that $\chi_{A \cup B} = \chi_A + \chi_B - \chi_{A \cap B}$ for any sets A, B

cases

Case 1 if $x \notin A \cup B \Rightarrow x \notin A$ & $x \notin B$

$0 = 0 + 0 - 0$



Case 2 if $x \in A - B$, similarly if $x \in B - A$ (case 3)

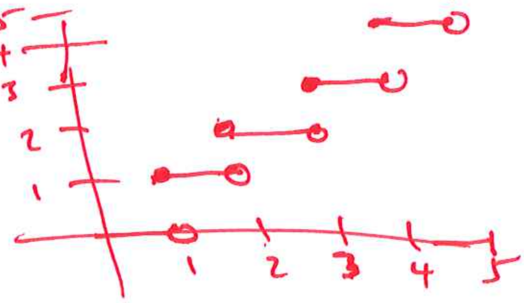
$\chi_{A \cup B} = 1$, $\chi_A = 1$, $\chi_B = 0$, $\chi_{A \cap B} = 0$

$\Rightarrow$ Equality holds

1 pts each

Case 4 $x \in A \cap B \Rightarrow x \in A, x \in B, x \in A \cup B$

$1 = 1 + 1 - 1$ ✓



3)(5points) a) Let $f(x) = [x]$

Find $f^{-1}\{2,3,4\}, f^{-1}[2,4].f^{-1}(2,3), f^{-1}(N), f(N),$

$f^{-1}\{2,3,4\} = f^{-1}(2) \cup f^{-1}(3) \cup f^{-1}(4)$
$= [2,3) \cup [3,4) \cup [4,5) = [2,5)$

$f^{-1}[2,4] = [2,5)$

$f^{-1}(2,3) = \phi$

$f^{-1}(\mathbb{N}) = [1,\infty)$

$f(\mathbb{N}) = \mathbb{N}$

b) (6 points) Let $f: X \to Y, B \subset Y$

1)Show that $f(f^{-1}(B)) \subset B$

2) Is $B \subset f(f^{-1}(B))$, prove or disprove.

1) let $y \in f(f^{-1}(B))$
$\Rightarrow \exists x \in f^{-1}(B)$ st $f(x) = y$
$\Rightarrow f(x) \in B$
$\Rightarrow y \in B$

from above
let $B = [1,\infty)$
$f^{-1}(B) = [1,\infty)$
$f(f^{-1}(B)) = \mathbb{N}$
But $B \not\subset \mathbb{N}$.

4) (6 points) a) Show that any subset of N is countable .

If X is finite $\Rightarrow X$ is countable by definition

If not, i.e X is infinite

lets define $f: \mathbb{N} \to X$ (Existed by Axiom of choice)

s.t $f(1)$ = least element in X , $f(1) \geq 1$

$f(2)$ = least element in $X \setminus \{f(1)\}$, $f(2) \geq 2$

.

$f(n)$ = least element in $X \setminus \{f(1), \dots, f(n-1)\}$ $f(n) \geq n$

f is 1-1, since the least is unique.

to show its onto, suppose not , $\exists z \in X \setminus f(\mathbb{N})$

$\Rightarrow z \in X / \{f(1), f(2), f(3), \dots\}$

But $f(z) \in \{f(1), f(2), \dots\} \} \Rightarrow f(z) \notin X \setminus \{f(1), f(2), \dots\}$

and $f(z) = \min\{X / f(1), f(2), \dots, f(z-1)\}$ & $f(z) \geq z$

But $z \notin \{f(1), f(2), \dots, f(z), \dots\} \Rightarrow f(z) < z$ X.

min $X \setminus f(1), \dots, f(z-1)\}$

b) (6 points)Show that if A,B are countable sets then AxB is countable set.

Since A,B are countable ,

$A = \{a_1, a_2, a_3, \dots$ $\qquad a_k = f(k)$, f is 1-1

$B = \{b_1, b_2, b_3, \dots$ $\qquad b_j = g(j)$, g is 1-1

$A \times B = \{ (a_1,b_1), (a_1,b_2), (a_1,b_3), (a_1,b_4) \dots$

(a_2,b_1) , $(a_2,b_2), (a_2,b_3)$ $(a_2,b_4) \dots$

(a_3,b_1) (a_3,b_2), $(a_3,b_3) \dots$

⋮

let $h(a_i,b_j) = \sum_{k=1}^{i+j-2} k + i$ $\qquad h$ is 1-1

5) (5 points) Let $f: X \to Y$. Prove that f is 1-1 if and only if $f(A) \cap f(B) = f(A \cap B)$

$1-1 \Leftrightarrow f(A) \cap f(B) = f(A \cap B)$ for every A, B

$(\Leftarrow)$ Let $f(x_1) = f(x_2)$ to show $x_1 = x_2$

or if $x_1 \neq x_2$ to show $f(x_1) \neq f(x_2)$

2pts) Since $x_1 \neq x_2 \Rightarrow \{x_1\} \cap \{x_2\} = \phi$

$\Rightarrow f(\{x_1\}) \cap f(\{x_2\}) = \phi$ by the assumption

$\Rightarrow f(x_1) \neq f(x_2)$

$(\Rightarrow)$ Suppose f is 1-1, to show

$f(A) \cap f(B) = f(A \cap B)$

$(\subseteq)$ let $y \in f(A) \cap f(B)$

2pts $\Rightarrow y \in f(A)$ & $y \in f(B)$

$\Rightarrow \exists x_1 \in A$ s.t $f(x_1) = y$ & $\exists x_2 \in B$ s.t $f(x_2) = y$.

But f is 1-1, $\Rightarrow x_1 = x_2 \in A \cap B$

$\Rightarrow y \in f(A \cap B)$

$(\supseteq)$ let $y \in f(A \cap B)$

$\Rightarrow \exists x \in A \cap B$ st $f(x) = y$

2pts $\Rightarrow x \in A$ & $x \in B$

$\Rightarrow f(x) \in f(A)$ and $f(x) \in f(B)$

$\Rightarrow f(x) \in f(A) \cap f(B)$