State Space approach

1) input variables $u(t)$
2) output variables $y(t)$
3) state variables $x(t)$

# of stat variables = # of integrations

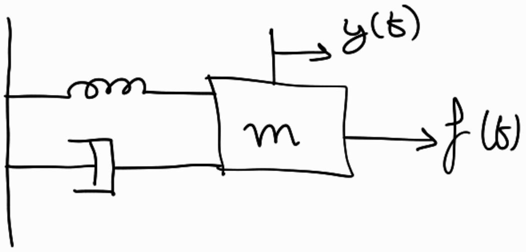
x_1, x_2 من الدرجة الثانية

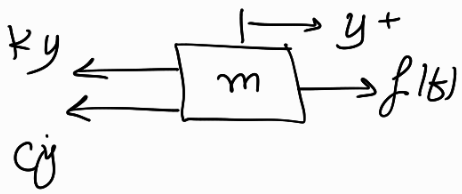
x_1, x_2, x_3 الثالثة ~ ~

(r inputs → you specify them

m outputs

n states





$$\sum f = m\ddot{y}$$

$$f(t) - ky - c\dot{y} = m\ddot{y}$$

$$m\ddot{y} + c\dot{y} + ky = f(t)$$

we need two state variables

$x_1 = y$ ← هاي دالة الإخراج الي بدنا نشتقه

$x_2 = \dot{y}$ وهاي مشتقتها

if there was x_3: $x_3 = \ddot{y}$

Derive them

$$\dot{x}_1 = \dot{y} = x_2$$

$$\dot{x}_2 = \ddot{y} = \frac{f(t)}{m} - \frac{c\dot{y}}{m} - \frac{ky}{m} = \frac{f(t)}{m} - \frac{c}{m}x_2 - \frac{k}{m}x_1$$

لازم اكتبها
بدلالة x_1, x_2

$$\underset{\substack{n\times 1\\ 2\times 1}}{\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}} = \underset{\substack{n\times n\\ 2\times 2}}{\overbrace{\begin{bmatrix}0 & 1\\ -\frac{k}{m} & -\frac{c}{m}\end{bmatrix}}^{A}} \underset{\substack{n\times 1\\ 2\times 1}}{\begin{bmatrix}x_1\\ x_2\end{bmatrix}} + \underset{\substack{n\times r\\ 2\times 1}}{\overbrace{\begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}}^{B}} \underset{\substack{r\times 1\\ 1\times 1}}{f(t)}$$

$$\underset{m\times 1}{\begin{bmatrix}y\\ \dot{y}\end{bmatrix}} = \underset{m\times n}{\overbrace{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}}^{C}} \underset{n\times 1}{\begin{bmatrix}x_1\\ x_2\end{bmatrix}} + \underset{m\times r}{\overbrace{\begin{bmatrix}0\\ 0\end{bmatrix}}^{D}} \underset{r\times 1}{f(t)}$$

From state space to transfer function

$$\frac{Y(s)}{U(s)} = ?$$

$$\dot{X}(t) = A\vec{x}(t) + B\vec{u}(t)$$
$$y(t) = C\vec{x}(t) + D\vec{u}(t)$$

A, B, C, D
constant matrices

$$\frac{Y(s)}{U(s)} =$$

$$sX(s) = A\,X(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

$$sX(s) - AX(s) = BU(s)$$

$$(sI - A)X(s) = BU(s)$$

$$X(s) = BU(s)(sI-A)^{-1}$$

$$Y(s) = C(sI-A)^{-1}BU(s) + DU(s)$$

$$\underset{(1\times 1)}{\frac{Y(s)}{U(s)}} = \underbrace{\underset{m\times n}{C}\,\underset{n\times n}{(sI-A)^{-1}}\,\underset{n\times r}{B}}_{\substack{m\times r\\ 1\times 1}} + \underset{\substack{m\times r\\ 1\times 1}}{D}$$

$$\boxed{\begin{array}{l} r \text{ inputs} \\ m \text{ outputs} \\ n \text{ states} \end{array}}$$

Equations of motion

$$\underbrace{\dot{e}(t)}_{\text{مشتقة ال states}} = A\,\underbrace{\vec{x}(t)}_{\text{المتغيرات تاعة ال system}} + B\,\underbrace{\vec{u}(t)}_{\text{inputs}}$$

$e_1, e_2, e_3, e_4 \leftarrow$ 4 states عندنا

$e_1 \to x_1$, $e_2 \to \dot{x}_1$, $e_3 \to x_2$, $e_4 \to \dot{x}_2$

$$\dot{e} = \begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \\ \dot{e}_4 \end{pmatrix}$$

عبارة اصفار ووحدات

ومحكي ← من ال motion equation

$$\underbrace{y(t)}_{\text{outputs i want}} = C\,\underbrace{\vec{x}(t)}_{\text{نفس الي فوق}} + D\,\underbrace{u(t)}_{\text{inputs of system}}$$

2 states

$$m_1 \ddot{x}_1 = F_1 - K_1 x_1 - C_1 \dot{x}_1 + K_2 (x_2 - x_1) + C_2 (\dot{x}_2 - \dot{x}_1)$$

2states

$$m_2 \ddot{x}_2 = F_2 - K_2 (x_2 - x_1) - C_2 (\dot{x}_2 - \dot{x}_1)$$

states

$$\boxed{\begin{array}{l} e_1 = x_1 \\ e_2 = \dot{x}_1 \\ e_3 = x_2 \\ e_4 = \dot{x}_2 \end{array}}$$

$$\dot{e}_1 = \dot{x}_1 = e_2$$
$$\dot{e}_3 = \dot{x}_2 = e_4$$

$$m_1 \dot{e}_3 = F_1 - K_1 e_1 - C_1 e_3 + K_2 (e_2 - e_1) + C_2 (e_4 - e_3)$$

$$m_2 \dot{e}_4 = F_2 - K_2 (e_2 - e_1) - C_2 (e_4 - e_3)$$

$$\dot e_3 = F_1\left(\cancel{\frac{1}{m_1}}\right) - \left(\cancel{\frac{K_1}{m_1}}\right) e_1 - \left(\cancel{\frac{C_1}{m_1}}\right) e_3 + \left(\frac{K_2}{m_1}\right)(e_2 - e_1) + \frac{C_2}{m_1}(e_4 - e_3)$$

$$= F_1\left(\frac{1}{m_1}\right) + e_1\left(\frac{-K_1}{m_1} - \frac{K_2}{m_1}\right) + e_3\left(\frac{-C_1}{m_1} - \frac{C_2}{m_1}\right) + \frac{K_2}{m_1}e_2 + \frac{C_2 e_4}{m_1}$$

$$\ddot X_1 = F_1\left(\frac{1}{m_1}\right) + X_1\left(\frac{-K_1}{m_1} - \frac{K_2}{m_1}\right) + \dot X_1\left(\frac{-C_1}{m_1} + \frac{-C_2}{m_1}\right) + \frac{K_2}{m_1}X_2 + \dot X_2\frac{C_2}{m_1}$$

$$\ddot X_2 = F_2\frac{1}{m_2} + X_1\left(\frac{K_2}{m_2}\right) + \dot X_1\left(\frac{C_2}{m_2}\right) + X_2\left(\frac{-K_2}{m_2}\right) + \dot X_2\left(\frac{-C_2}{m_2}\right)$$

input matrix

$$\begin{pmatrix} \dot e_1 \\ \dot e_2 \\ \dot e_3 \\ \dot e_4 \end{pmatrix} = \underset{\substack{n\times 1 \\ 4\times 1}}{\begin{pmatrix} \dot X_1 \\ \ddot X_1 \\ \dot X_2 \\ \ddot X_2 \end{pmatrix}} = \underset{\substack{4\times 4 \\ A}}{\begin{pmatrix} 0 & 1 & 0 & 0 \\ \left(\frac{-K_1}{m_1} - \frac{K_2}{m_1}\right) & \left(\frac{-C_1}{m_1} - \frac{C_2}{m_2}\right) & \frac{K_2}{m_1} & \frac{C_2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{K_2}{m_2} & \frac{C_2}{m_2} & \frac{-K_2}{m_2} & \frac{-C_2}{m_2} \end{pmatrix}} \underset{4\times 1}{\begin{pmatrix} X_1 \\ \dot X_1 \\ X_2 \\ \dot X_2 \end{pmatrix}}$$

$$+ \underset{\substack{n\times r \\ 4\times 2}}{\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{pmatrix}} \underset{\substack{r\times 1 \\ 2\times 1}}{\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}}$$

$e_1 = X_1$

$e_2 = \dot X_1$

$\dot e_1 = \dot X_1 = e_2$

$\dot e_3 = \dot X_2 = e_4$

output

$$\underset{6\times 1}{\begin{pmatrix} X_1 \\ \dot X_1 \\ \ddot X_1 \\ X_2 \\ \dot X_2 \\ \ddot X_2 \end{pmatrix}} = \underset{6\times 4}{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-K_1}{m_1}\ \frac{-K_2}{m_2} & \frac{-K_1}{m_1}\ \frac{-K_2}{m_2} & \frac{K_2}{m_1} & \frac{C_2}{m_1} \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 \\ \frac{K_2}{m_2} & \frac{C_2}{m_2} & \frac{-K_2}{m_2} & \frac{-C_2}{m_2} \end{pmatrix}} \underset{4\times 1}{\begin{pmatrix} X_1 \\ \dot X_1 \\ X_2 \\ \dot X_2 \end{pmatrix}} + \underset{6\times 2}{\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{pmatrix}} \underset{2\times 1}{\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}}$$

Assuming

$$K_1 = 2, \quad C_1 = 3 \quad m_1 = 2$$
$$K_2 = 3 \quad C_2 = 4 \quad m_2 = 4$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2.5 & -3.5 & 1.5 & 2 \\ 0 & 0 & 0 & 1 \\ 0.75 & 1 & -0.75 & -1 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0.5 & 0 \\ 0 & 0.25 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2.5 & -3.5 & 1.5 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.75 & 1 & -0.75 & -1 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0.5 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0.25 \end{pmatrix}$$