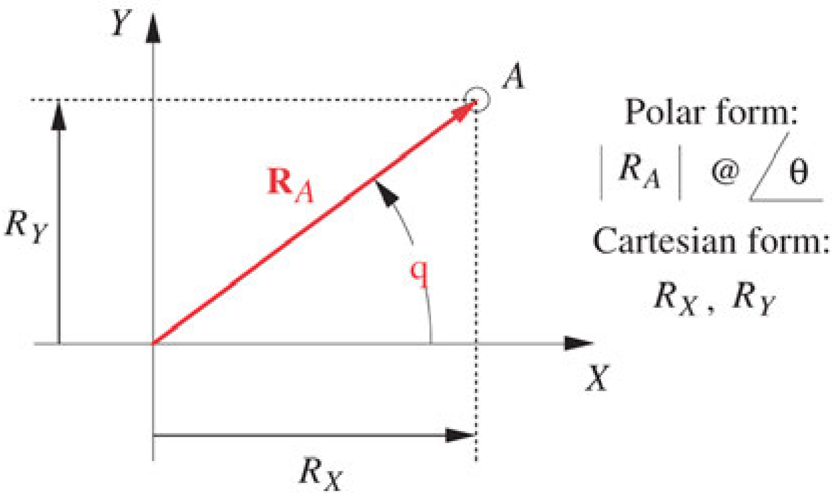
Position Analysis

2015-03-02

Position

- The position of a point in the plane can be defined by the use of a **position vector**
  - Cartesian coordinates
  - Polar coordinates



(a) Global coordinate system XY

Each form is directly convertible into the other by:

the Pythagorean theorem:

$$R_A = \sqrt{R_X^2 + R_X^2}$$

and trigonometry:

$$\theta = \arctan\left(\frac{R_Y}{R_X}\right)$$

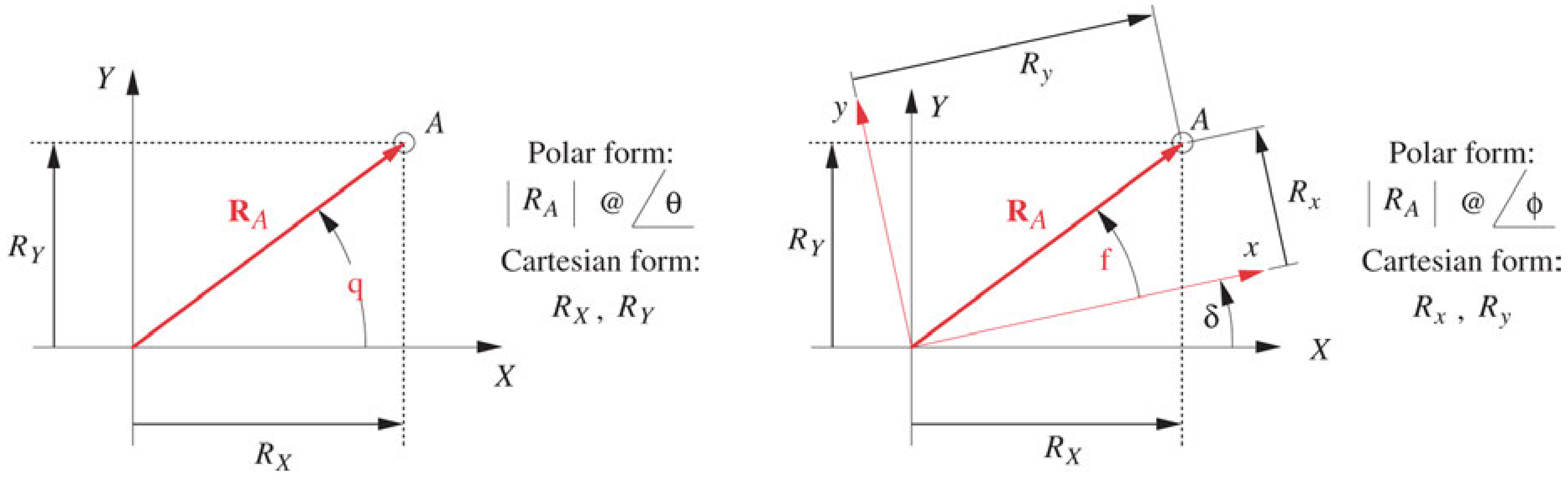
REVISION

Position

Coordinate Transformation

- If the position of point *A* is expressed in the local *xy* system, and it is desired to transform its coordinates to the global *XY* system, the equations are:

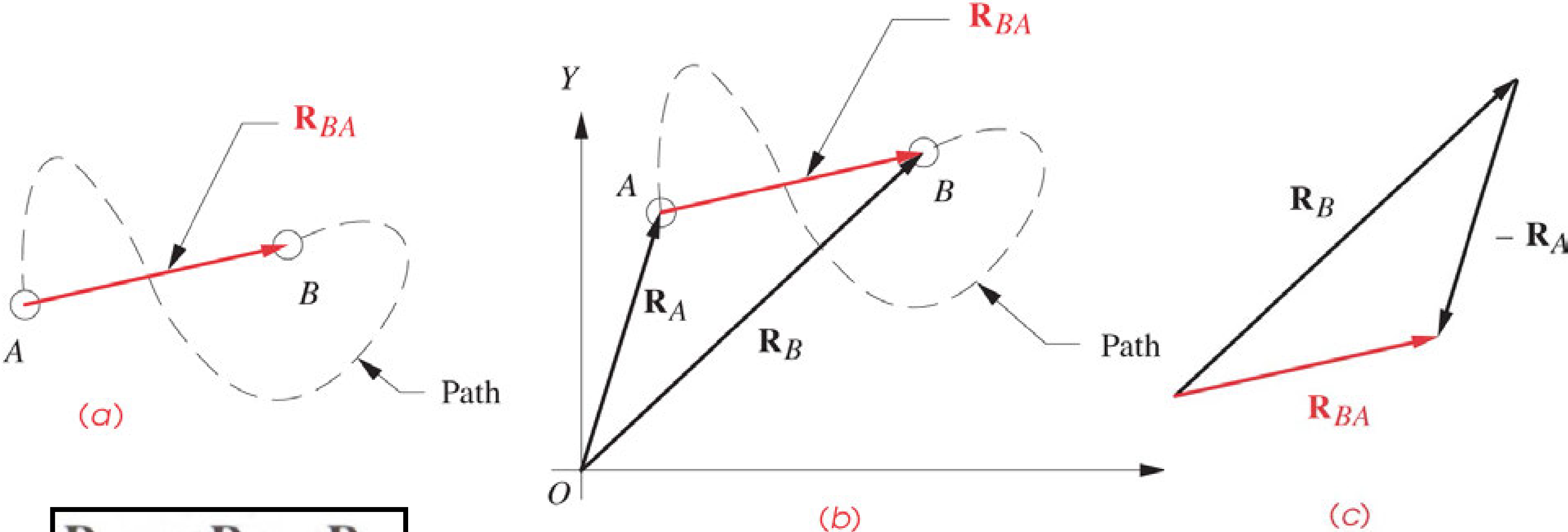
$$R_X = R_x \cos\delta - R_y \sin\delta$$
$$R_Y = R_x \sin\delta + R_y \cos\delta$$



(a) Global coordinate system *XY* (b) Local coordinate system *xy*

Displacement

- **Displacement of a point:** is the change in its position and can be defined as the straight-line distance between the initial and final position of a point which has moved in the reference frame.



$$\mathbf{R}_{BA} = \mathbf{R}_B - \mathbf{R}_A$$

REVISION

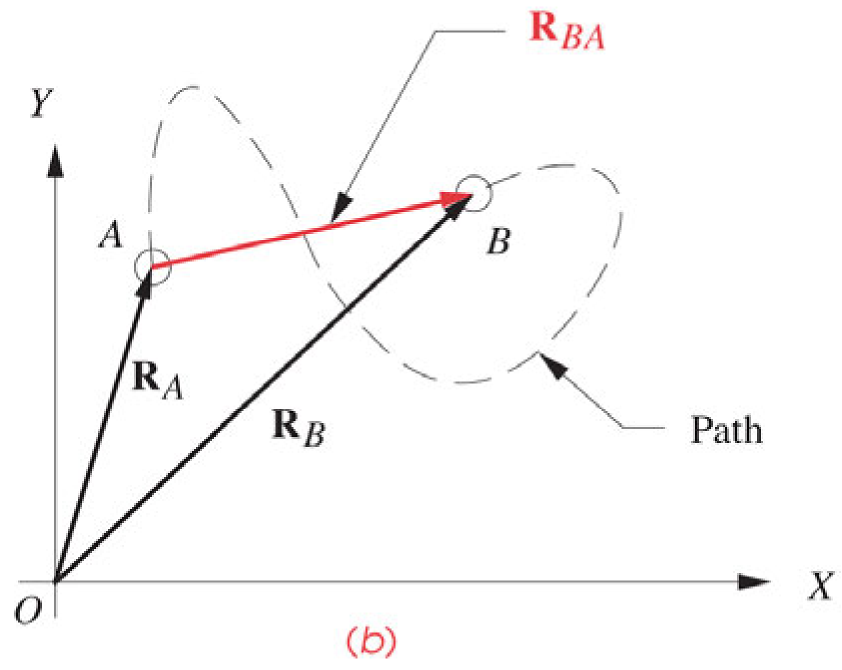
Displacement

- **Position difference equation:** The position of B with respect to A is equal to the (absolute) position of B minus the (absolute) position of A

$$\mathbf{R}_{BA} = \mathbf{R}_B - \mathbf{R}_A$$

OR

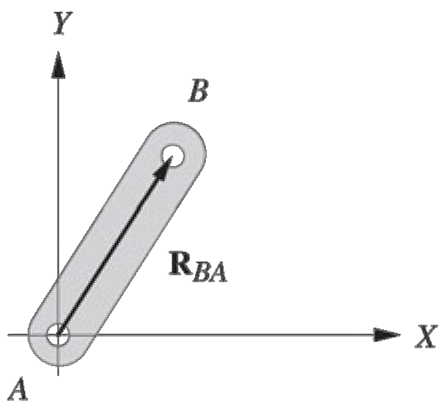
$$\mathbf{R}_{BA} = \mathbf{R}_{BO} - \mathbf{R}_{AO}$$

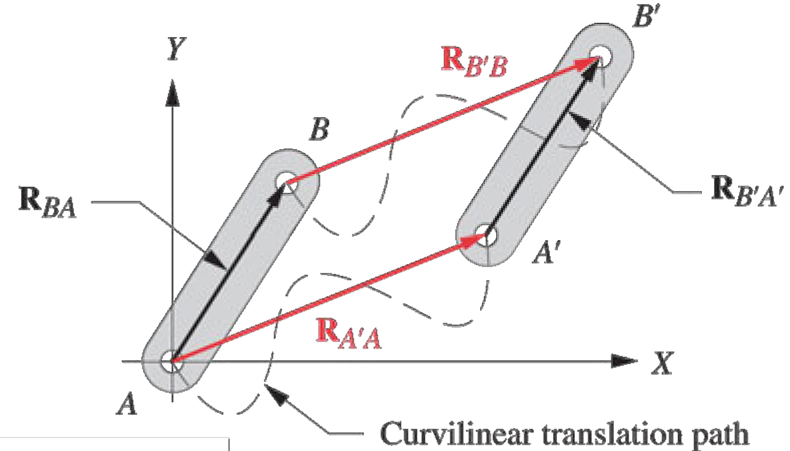


Translation, Rotation, and Complex motion

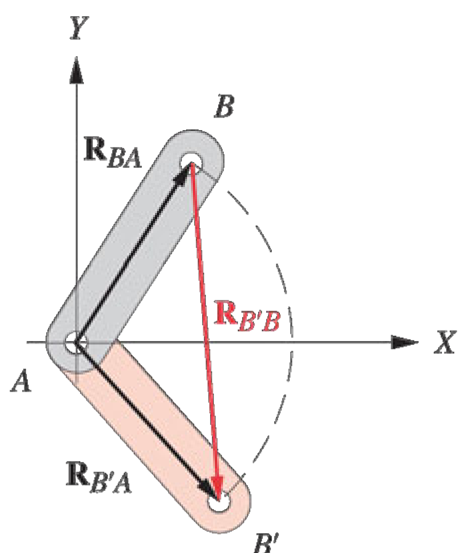
REVISION

- Translation





- *Rotation*



Translation, Rotation, and Complex motion

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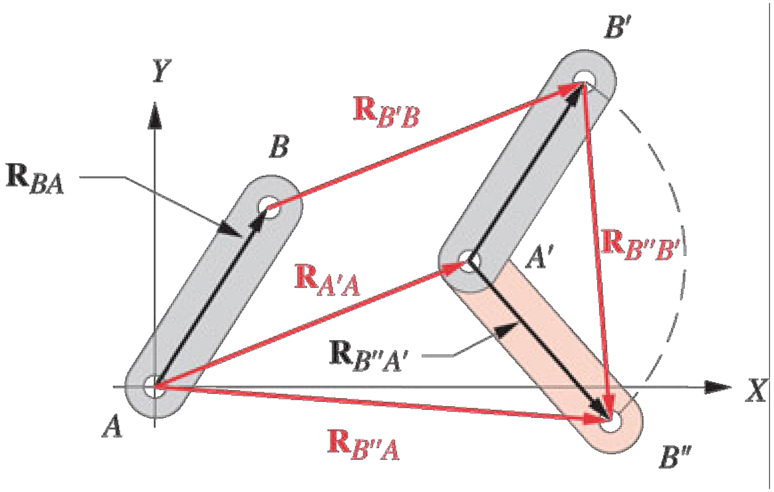
- Complex motion
  - *Total displacement =translation component + rotation component*



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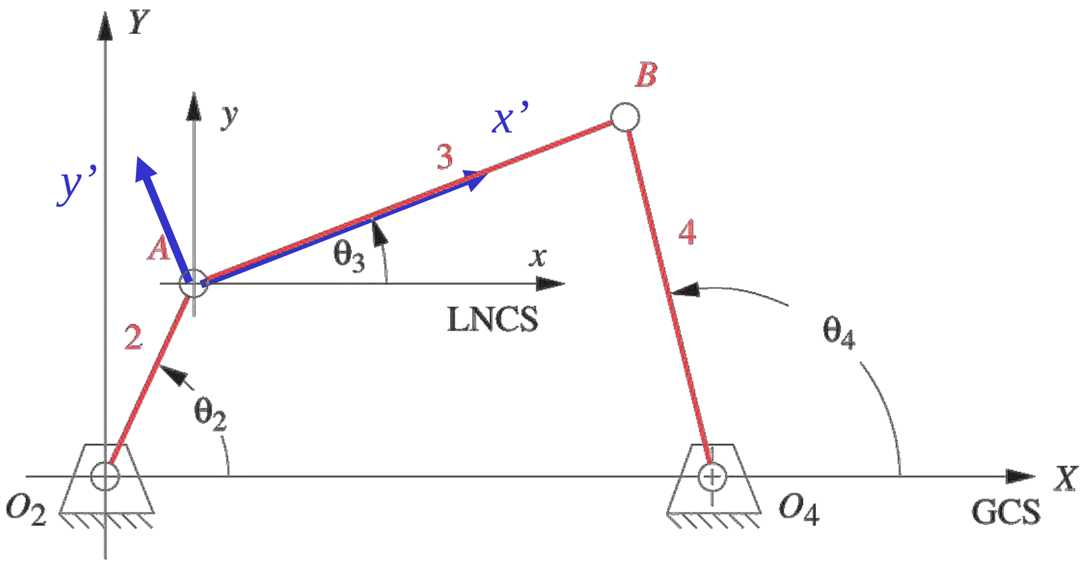
Coordinate Systems



Coordinate Systems:

GCS = Global Coordinate System, (*X*, *Y*)

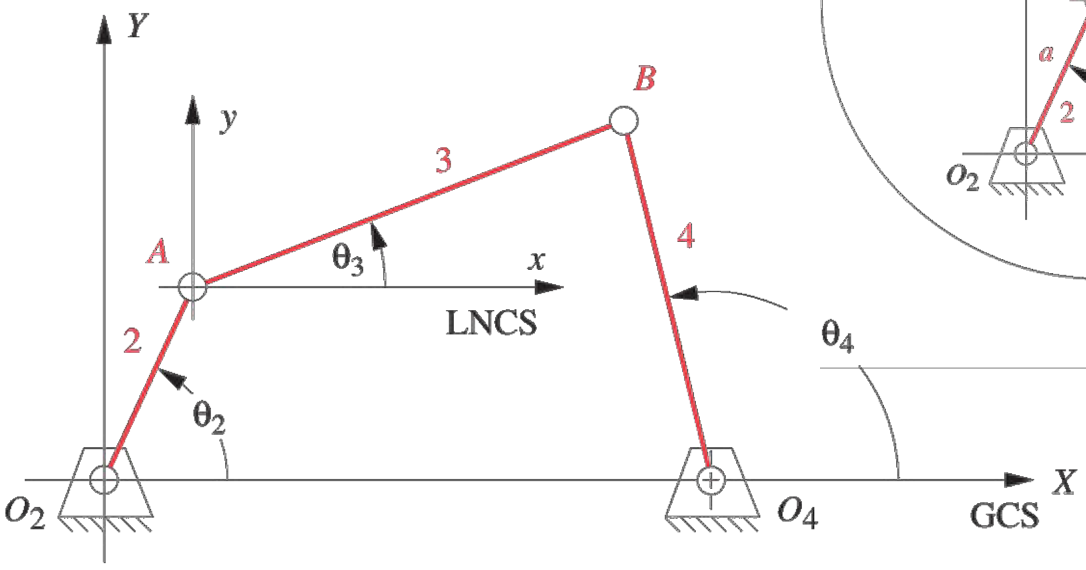
LNCS = Local Non-Rotating Coordinate System , (*x*, *y*)

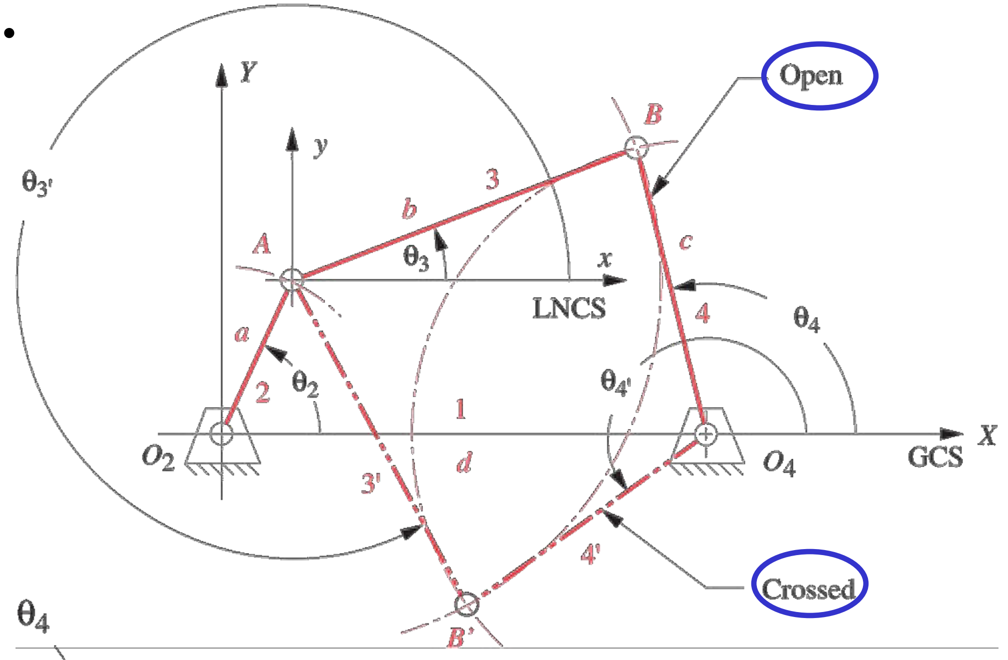
LRCS = Local Rotating Coordinate System , (*x'*, *y'*)

Graphical Position Analysis

- The graphical analysis of this problem is trivial and can be done using only high school geometry.

Position at θ3





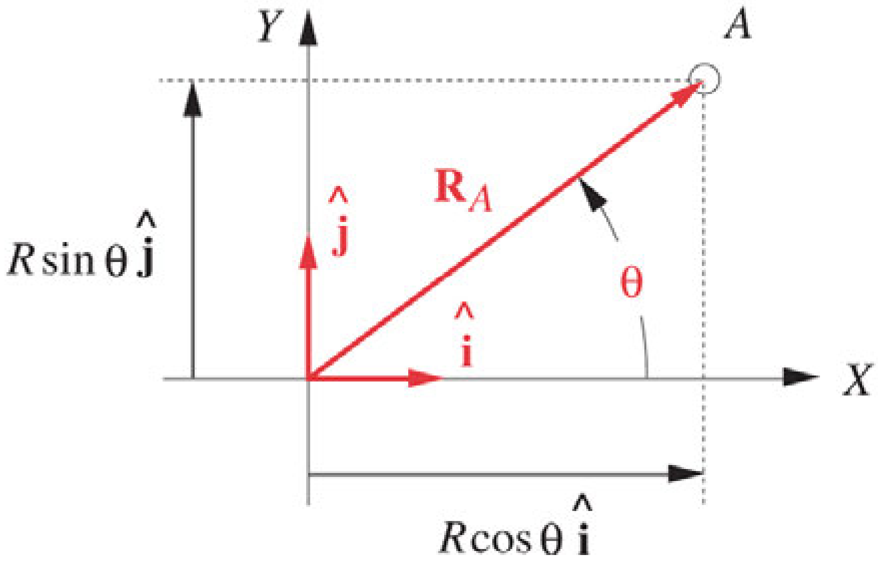
Position at θ3, θ3'

REVISION

Algebraic Position Analysis

- Complex Numbers as Vectors

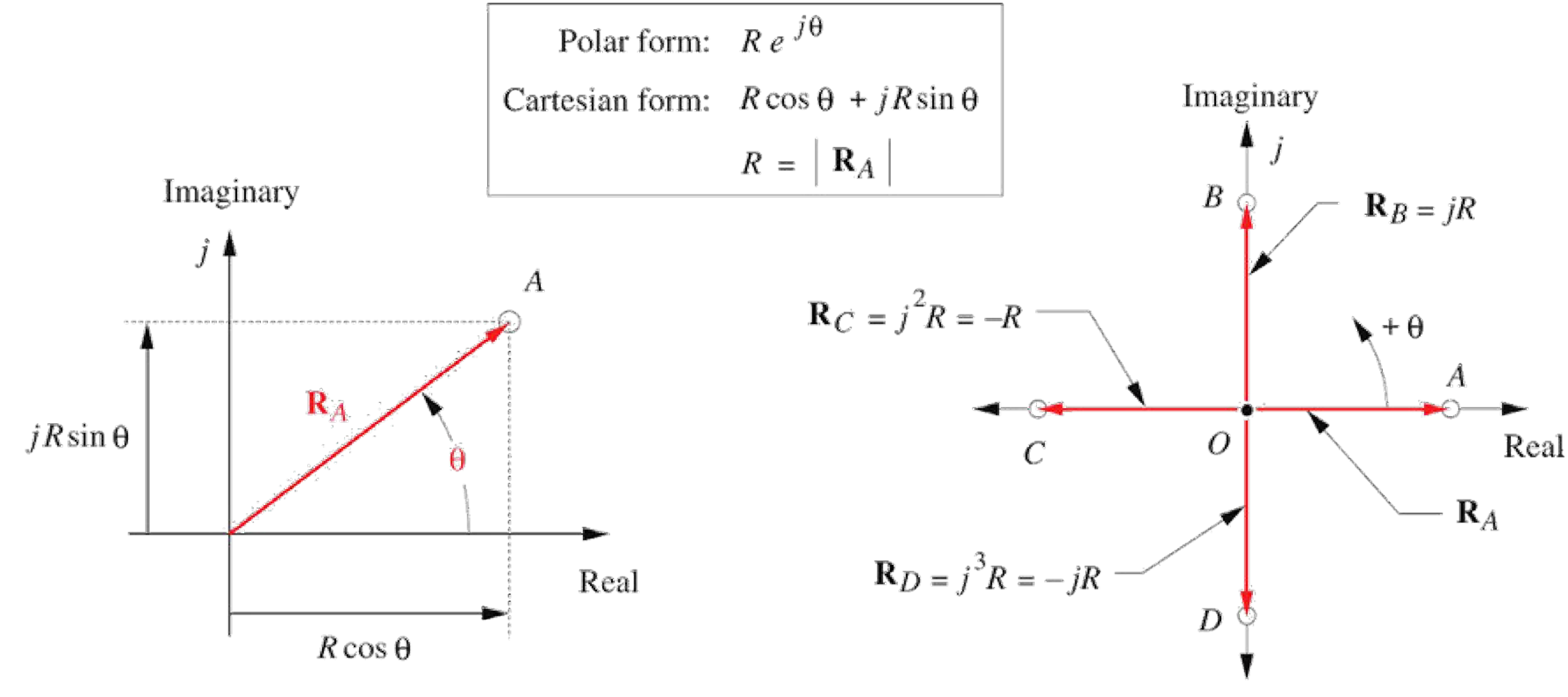
  - Remember the Euler identity: $e^{\pm j\theta} = \cos\theta \pm j\sin\theta$



Polar form	Cartesian form
$R@\angle\theta$	$r\cos\theta\,\hat{\mathbf{i}} + r\sin\theta\,\hat{\mathbf{j}}$
$re^{j\theta}$	$r\cos\theta + jr\sin\theta$

Algebraic Position Analysis

- Complex Numbers as Vectors



(a) Complex number representation of a position vector

(b) Vector rotations in the complex plane

We use complex number notation for vectors to develop and derive the equations for position, velocity, and acceleration of linkages.

Algebraic Position Analysis

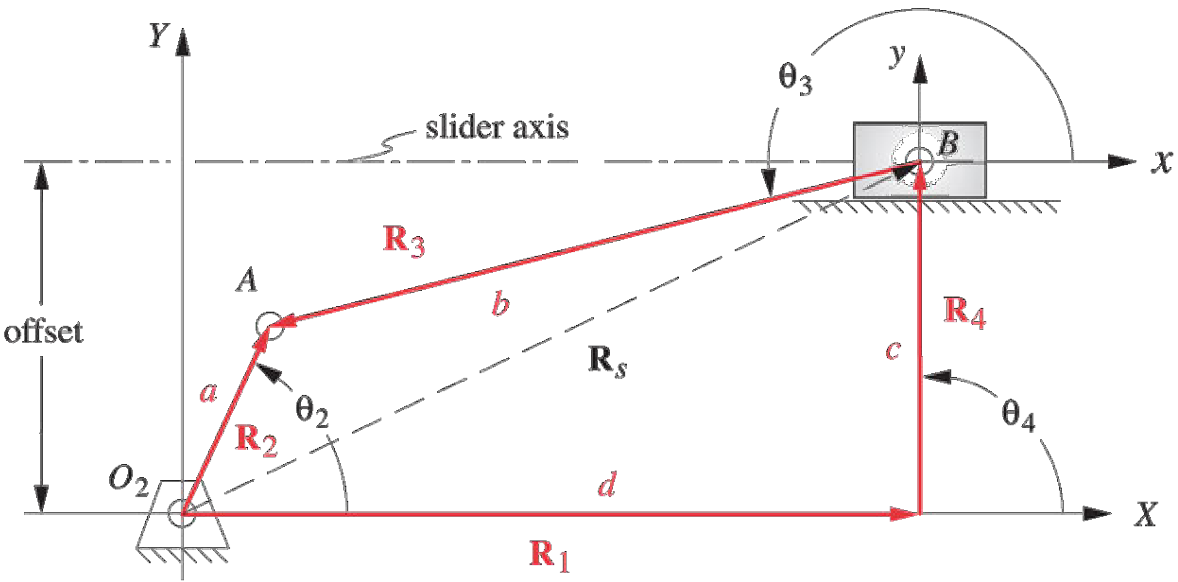
<u>The Vector loop Equation for a Slider-Crank</u>

1. Write the vector loop equation:

$$\mathbf{R}_2 - \mathbf{R}_3 - \mathbf{R}_4 - \mathbf{R}_1 = 0$$

2. We substitute the complex number notation for each position vector:

$$a e^{j\theta_2} - b e^{j\theta_3} - c e^{j\theta_4} - d e^{j\theta_1} = 0$$



Algebraic Position Analysis

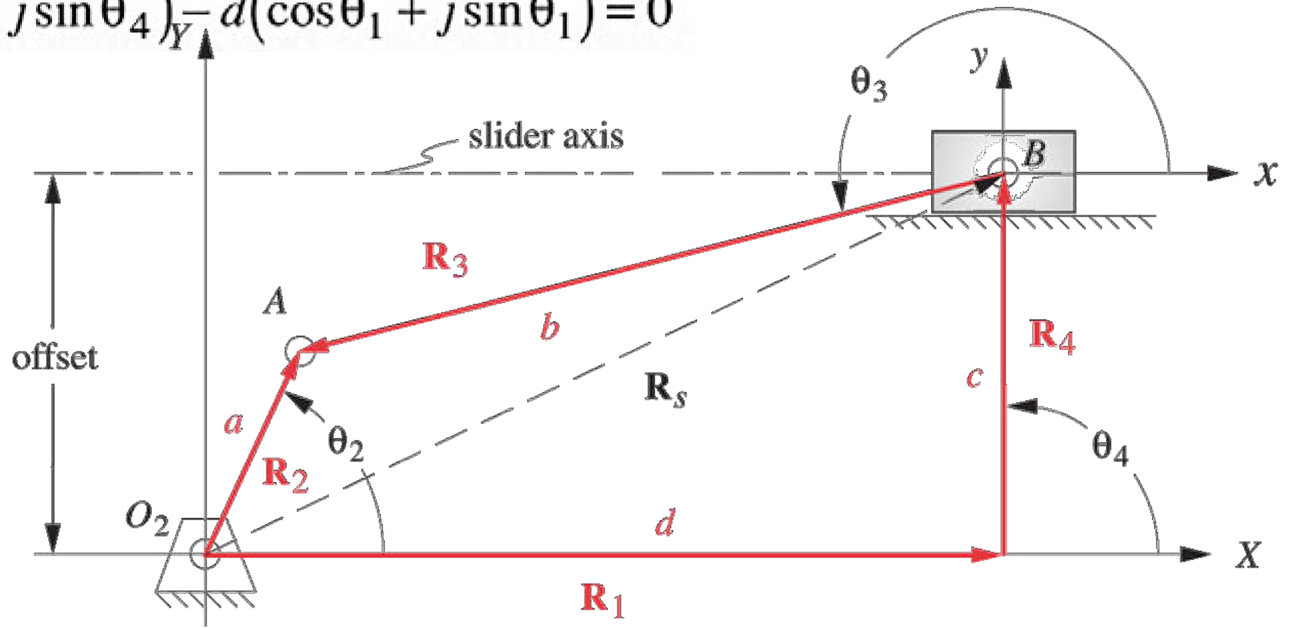
<u>The Fourbar Slider-Crank Position Solution</u>

$$\mathbf{R}_2 - \mathbf{R}_3 - \mathbf{R}_4 - \mathbf{R}_1 = 0$$

$$a e^{j\theta_2} - b e^{j\theta_3} - c e^{j\theta_4} - d e^{j\theta_1} = 0$$

√ Input √ ? √ √ ? √

$$a(\cos\theta_2 + j\sin\theta_2) - b(\cos\theta_3 + j\sin\theta_3)$$
$$- c(\cos\theta_4 + j\sin\theta_4) - d(\cos\theta_1 + j\sin\theta_1) = 0$$



Algebraic Position Analysis

The Fourbar Slider-Crank Position Solution

Separate the real and imaginary components:

real part (x component):

$$a\cos\theta_2 - b\cos\theta_3 - c\cos\theta_4 - d\cos\theta_1 = 0$$

but: $\theta_1 = 0$, so:

$$a\cos\theta_2 - b\cos\theta_3 - c\cos\theta_4 - d = 0$$

imaginary part (y component):

$$ja\sin\theta_2 - jb\sin\theta_3 - jc\sin\theta_4 - jd\sin\theta_1 = 0$$

but: $\theta_1 = 0$, and the j's divide out, so:

$$a\sin\theta_2 - b\sin\theta_3 - c\sin\theta_4 = 0$$

Algebraic Position Analysis

The Fourbar Slider-Crank Position Solution

From $$a\cos\theta_2 - b\cos\theta_3 - c\cos\theta_4 - d = 0$$

? **?**

and $$a\sin\theta_2 - b\sin\theta_3 - c\sin\theta_4 = 0$$

The solution is:

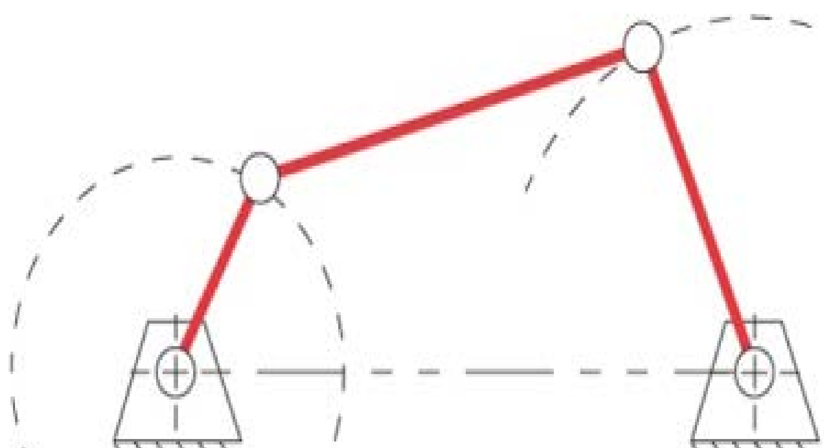
$$\theta_{3_1} = \arcsin\left(\frac{a\sin\theta_2 - c}{b}\right)$$

$$d = a\cos\theta_2 - b\cos\theta_3$$

Algebraic Position Analysis

<u>The Vector loop Equation for a Fourbar linkage</u>

1. Write the vector loop equation:

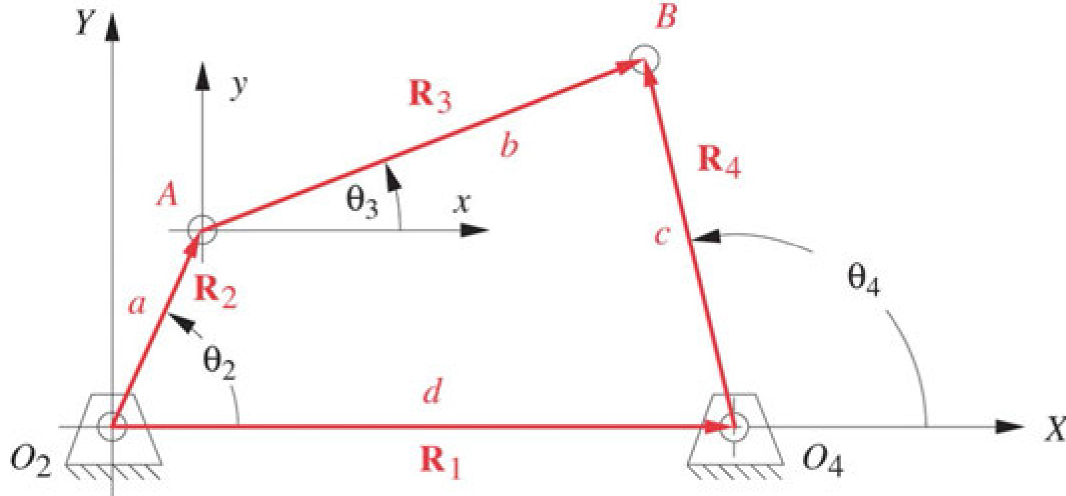


Algebraic Position Analysis

<u>The Vector loop Equation for a Fourbar linkage</u>

1. Write the vector loop equation: $\mathbf{R}_2 + \mathbf{R}_3 - \mathbf{R}_4 - \mathbf{R}_1 = 0$

2. We substitute the complex number notation for each position vector: $a e^{j\theta_2} + b e^{j\theta_3} - c e^{j\theta_4} - d e^{j\theta_1} = 0$



Algebraic Position Analysis

<u>The Vector loop Equation for a Fourbar linkage</u>

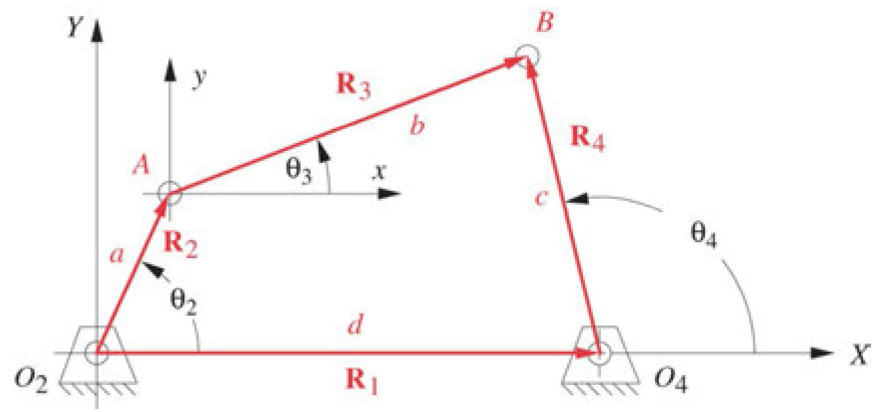
$$a e^{j\theta_2} + b e^{j\theta_3} - c e^{j\theta_4} - d e^{j\theta_1} = 0$$

√ √ √ ? √ ? √ √

Input

=0

We have two unknowns θ2, θ3. We need two equations.



Algebraic Position Analysis

The Vector loop Equation for a Fourbar linkage

Subsituting Euler equivalents for the $e^{j\theta}$ into

$$a e^{j\theta_2} + b e^{j\theta_3} - c e^{j\theta_4} - d e^{j\theta_1} = 0$$

we get:

$$a(\cos\theta_2 + j\sin\theta_2) + b(\cos\theta_3 + j\sin\theta_3) - c(\cos\theta_4 + j\sin\theta_4) - d(\cos\theta_1 + j\sin\theta_1) = 0$$

This equation can now be separated into its real and imaginary parts and each set to zero.

real part (x component):

$$a\cos\theta_2 + b\cos\theta_3 - c\cos\theta_4 - d\cos\theta_1 = 0$$

but: $\theta_1 = 0$, so: (4.6a)

$$a\cos\theta_2 + b\cos\theta_3 - c\cos\theta_4 - d = 0$$

Algebraic Position Analysis

The Vector loop Equation for a Fourbar linkage

real part (x component):

$$a\cos\theta_2 + b\cos\theta_3 - c\cos\theta_4 - d = 0$$

imaginary part (y component):

$$ja\sin\theta_2 + jb\sin\theta_3 - jc\sin\theta_4 - jd\sin\theta_1 = 0$$

but: $\theta_1 = 0$, and the $j's$ divide out, so: (4.6b)

$$a\sin\theta_2 + b\sin\theta_3 - c\sin\theta_4 = 0$$

We will isolate θ_3 and solve for θ_4 in this example.

$$b\cos\theta_3 = -a\cos\theta_2 + c\cos\theta_4 + d$$
$$b\sin\theta_3 = -a\sin\theta_2 + c\sin\theta_4$$

Algebraic Position Analysis

<u>The Vector loop Equation for a Fourbar linkage</u>

$$b\cos\theta_3 = -a\cos\theta_2 + c\cos\theta_4 + d$$

$$b\sin\theta_3 = -a\sin\theta_2 + c\sin\theta_4$$

Now square both sides of equations 4.6c and 4.6d and add them:

$$b^2\left(\sin^2\theta_3 + \cos^2\theta_3\right) = \left(-a\sin\theta_2 + c\sin\theta_4\right)^2 + \left(-a\cos\theta_2 + c\cos\theta_4 + d\right)^2$$

=1

The right side of this expression must now be expanded and terms collected.

$$b^2 = a^2 + c^2 + d^2 - 2ad\cos\theta_2 + 2cd\cos\theta_4 - 2ac\left(\sin\theta_2\sin\theta_4 + \cos\theta_2\cos\theta_4\right)$$

Algebraic Position Analysis

The Vector loop Equation for a Fourbar linkage

$$b^2 = a^2 + c^2 + d^2 - 2ad\cos\theta_2 + 2cd\cos\theta_4 - 2ac(\sin\theta_2\sin\theta_4 + \cos\theta_2\cos\theta_4)$$

To further simplify this expression, the constants K_1, K_2, and K_3 are defined

$$K_1 = \frac{d}{a} \qquad K_2 = \frac{d}{c} \qquad K_3 = \frac{a^2 - b^2 + c^2 + d^2}{2ac} \tag{4.8a}$$

and:

$$K_1\cos\theta_4 - K_2\cos\theta_2 + K_3 = \cos\theta_2\cos\theta_4 + \sin\theta_2\sin\theta_4 \tag{4.8b}$$

If we substitute the identity $\cos(\theta_2 - \theta_4) = \cos\theta_2\cos\theta_4 + \sin\theta_2\sin\theta_4$, we get the

$$K_1\cos\theta_4 - K_2\cos\theta_2 + K_3 = \cos(\theta_2 - \theta_4) \tag{4.8c}$$

Freudenstein's equation

Algebraic Position Analysis

<u>The Vector loop Equation for a Fourbar linkage</u>

$$K_1 \cos\theta_4 - K_2 \cos\theta_2 + K_3 = \cos(\theta_2 - \theta_4) \tag{4.8c}$$

Using the trigonometric rule

$$\sin\theta_4 = \frac{2\tan\left(\frac{\theta_4}{2}\right)}{1+\tan^2\left(\frac{\theta_4}{2}\right)}; \qquad \cos\theta_4 = \frac{1-\tan^2\left(\frac{\theta_4}{2}\right)}{1+\tan^2\left(\frac{\theta_4}{2}\right)}$$

We get

$$A\tan^2\left(\frac{\theta_4}{2}\right) + B\tan\left(\frac{\theta_4}{2}\right) + C = 0$$

$$A = \cos\theta_2 - K_1 - K_2\cos\theta_2 + K_3$$
$$B = -2\sin\theta_2$$
$$C = K_1 - (K_2+1)\cos\theta_2 + K_3$$

The solution is:

$$\tan\left(\frac{\theta_4}{2}\right) = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$\theta_{4_{1,2}} = 2\arctan\left(\frac{-B \pm \sqrt{B^2 - 4AC}}{2A}\right)$$

Algebraic Position Analysis

The Vector loop Equation for a Fourbar linkage

Back to:

$$b\cos\theta_3 = -a\cos\theta_2 + c\cos\theta_4 + d$$
$$b\sin\theta_3 = -a\sin\theta_2 + c\sin\theta_4$$

$$c\cos\theta_4 = a\cos\theta_2 + b\cos\theta_3 - d$$
$$c\sin\theta_4 = a\sin\theta_2 + b\sin\theta_3$$

Squaring and adding:

$$K_1\cos\theta_3 + K_4\cos\theta_2 + K_5 = \cos\theta_2\cos\theta_3 + \sin\theta_2\sin\theta_3$$

$$K_4 = \frac{d}{b}; \qquad K_5 = \frac{c^2 - d^2 - a^2 - b^2}{2ab}$$

Algebraic Position Analysis

The Vector loop Equation for a Fourbar linkage

$$K_1 \cos\theta_3 + K_4 \cos\theta_2 + K_5 = \cos\theta_2 \cos\theta_3 + \sin\theta_2 \sin\theta_3$$

similarly

$$D \tan^2\left(\frac{\theta_3}{2}\right) + E \tan\left(\frac{\theta_3}{2}\right) + F = 0$$

$$D = \cos\theta_2 - K_1 + K_4 \cos\theta_2 + K_5$$

$$E = -2\sin\theta_2$$

$$F = K_1 + (K_4 - 1)\cos\theta_2 + K_5$$

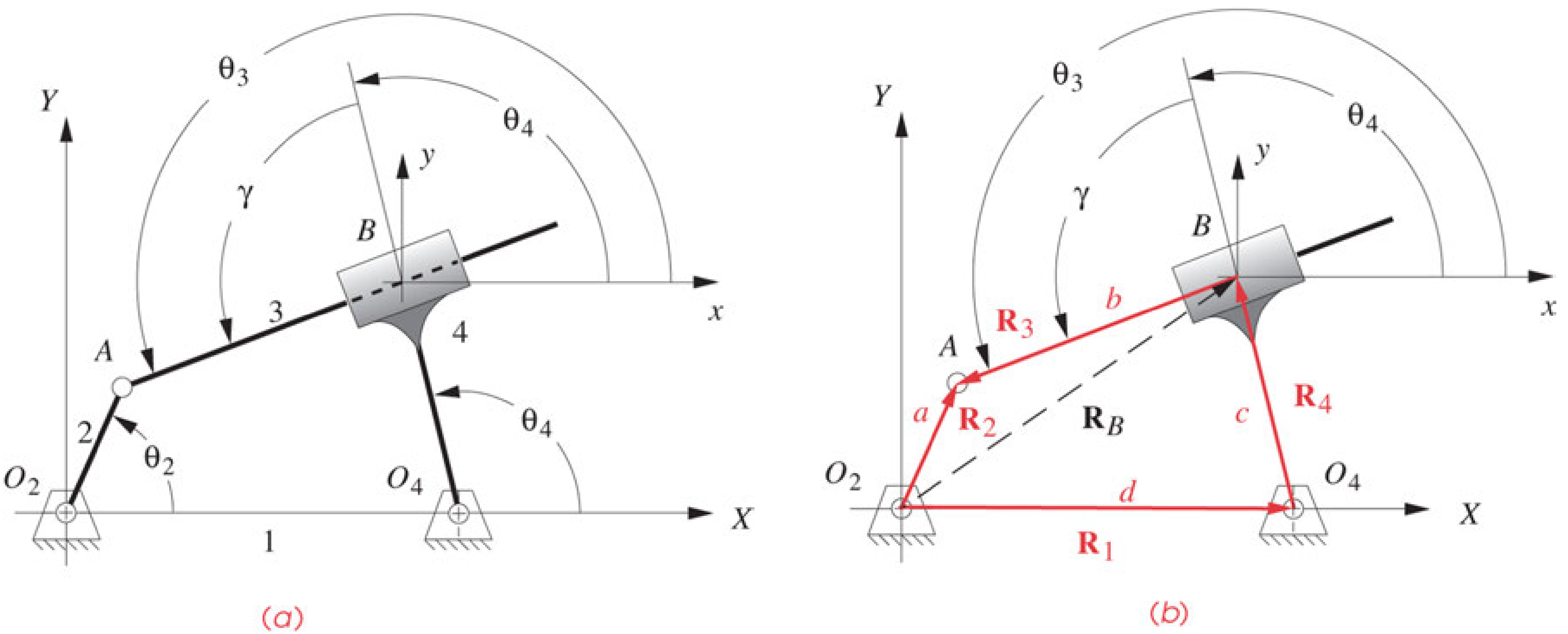
The solution is:

$$\theta_{3_{1,2}} = 2\arctan\left(\frac{-E \pm \sqrt{E^2 - 4DF}}{2D}\right)$$

Algebraic Position Analysis

Inverted Slider-Crank Position Solution

- This is inversion #3 of the common fourbar slider-crank linkage in which the sliding joint is between links 3 and 4 at point *B*. This is shown as an offset slider-crank mechanism. The slider block has pure rotation with its center offset from the slide



Algebraic Position Analysis

Inverted Slider-Crank Position Solution

$R_B = R_2 - R_3$

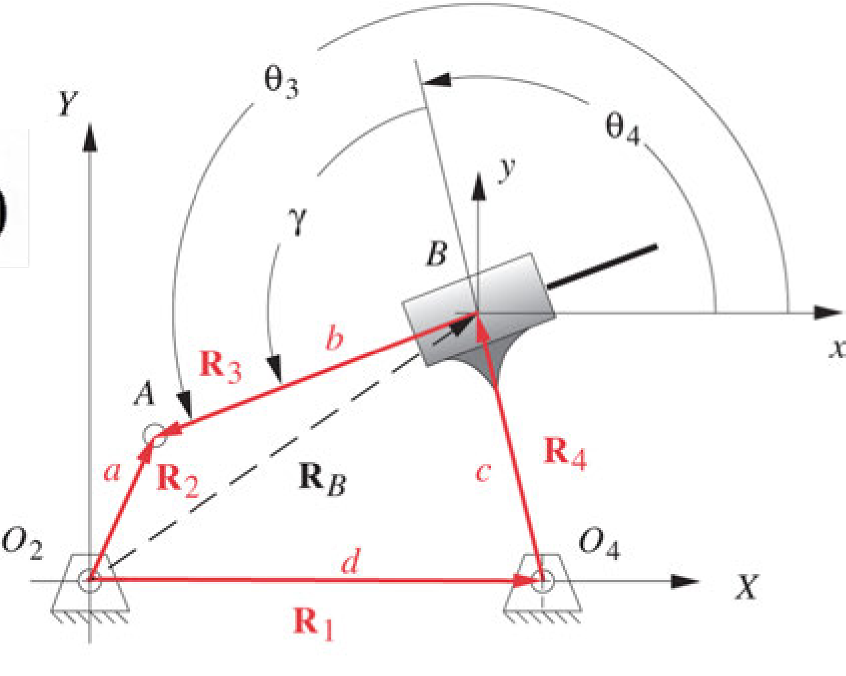
$$\mathbf{R}_2 - \mathbf{R}_3 - \mathbf{R}_4 - \mathbf{R}_1 = 0$$

$$a e^{j\theta_2} - b e^{j\theta_3} - c e^{j\theta_4} - d e^{j\theta_1} = 0$$

√ (a) Input (θ_2) ? (b) ? (θ_3) √ (c) ? (θ_4) √ (d) √ (θ_1)

We need 3rd equation

$$\theta_3 = \theta_4 \pm \gamma$$



(b)

Algebraic Position Analysis

Inverted Slider-Crank Position Solution

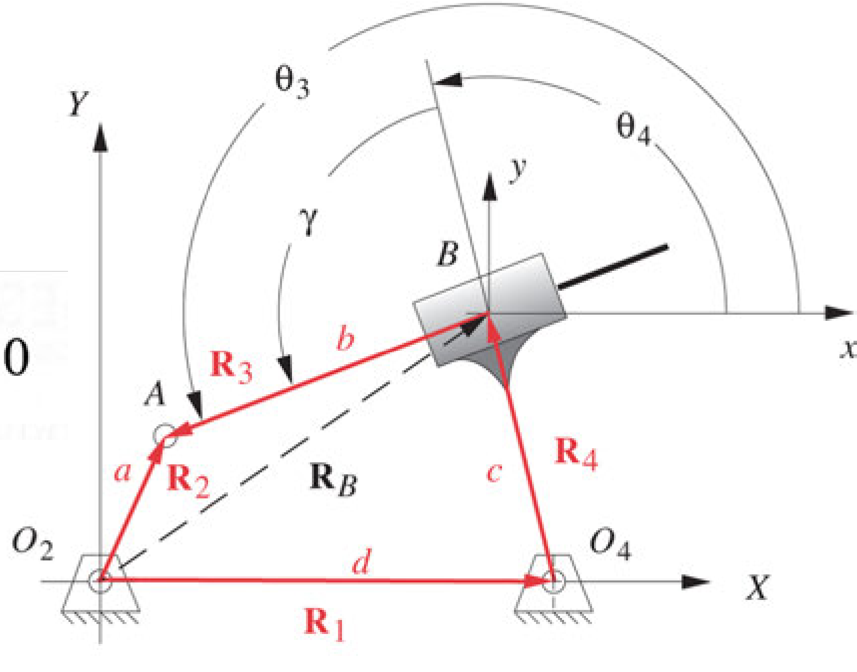
$$a\left(\cos\theta_2 + j\sin\theta_2\right) - b\left(\cos\theta_3 + j\sin\theta_3\right) - c\left(\cos\theta_4 + j\sin\theta_4\right) - d\left(\cos\theta_1 + j\sin\theta_1\right) = 0$$

$$a\cos\theta_2 - b\cos\theta_3 - c\cos\theta_4 - d = 0$$

$$a\sin\theta_2 - b\sin\theta_3 - c\sin\theta_4 = 0$$

$$b = \frac{a\sin\theta_2 - c\sin\theta_4}{\sin\theta_3}$$

$$a\cos\theta_2 - \frac{a\sin\theta_2 - c\sin\theta_4}{\sin\theta_3}\cos\theta_3 - c\cos\theta_4 - d = 0$$



(b)

Algebraic Position Analysis

Inverted Slider-Crank Position Solution

$$P\sin\theta_4 + Q\cos\theta_4 + R = 0$$

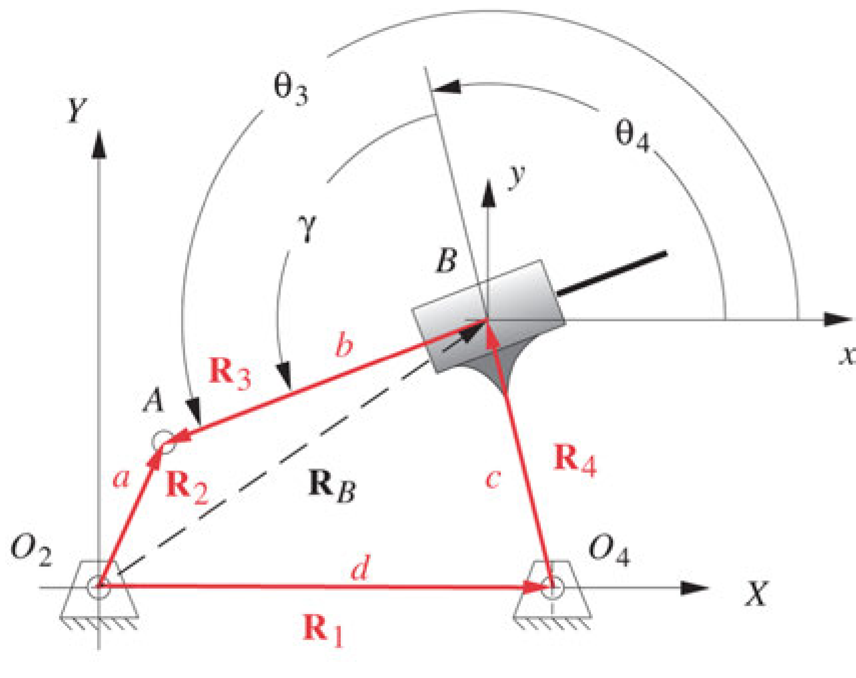
$$P = a\sin\theta_2 \sin\gamma + (a\cos\theta_2 - d)\cos\gamma$$

$$Q = -a\sin\theta_2 \cos\gamma + (a\cos\theta_2 - d)\sin\gamma$$

$$R = -c\sin\gamma$$

$$\theta_{4_{1,2}} = 2\arctan\left(\frac{-T \pm \sqrt{T^2 - 4SU}}{2S}\right)$$

$$S = R - Q; \qquad T = 2P; \qquad U = Q + R$$

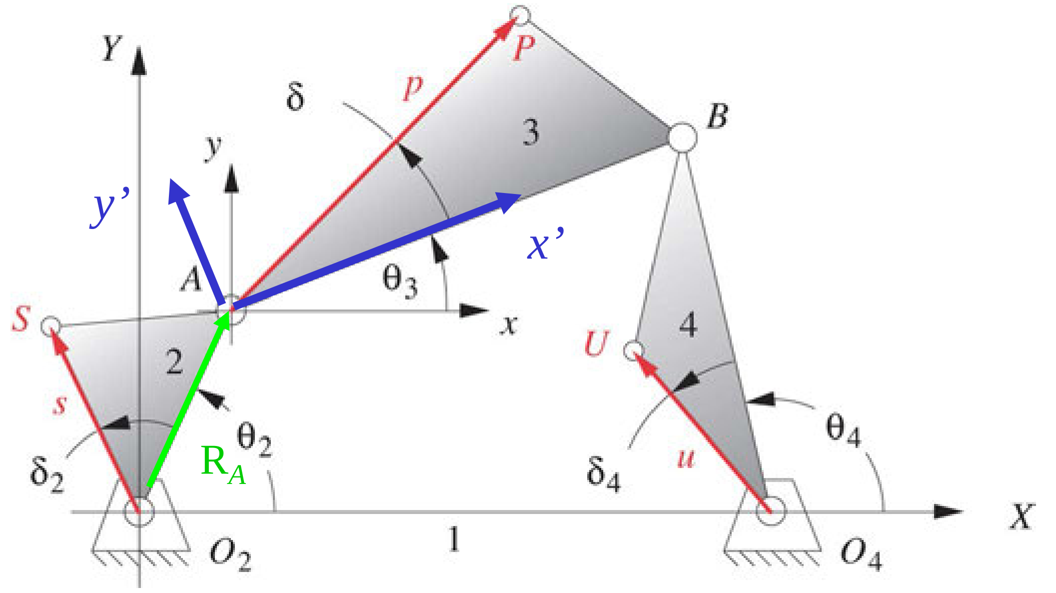


(b)

Position of a Point on a Link

Once the angles of all the links are found, it is simple and straightforward to define and calculate the position of any point on any link for any input position of the linkage.

Example: 4-bar linkage



$$\mathbf{R}_{SO_2} = \mathbf{R}_S = se^{j(\theta_2 + \delta_2)}$$

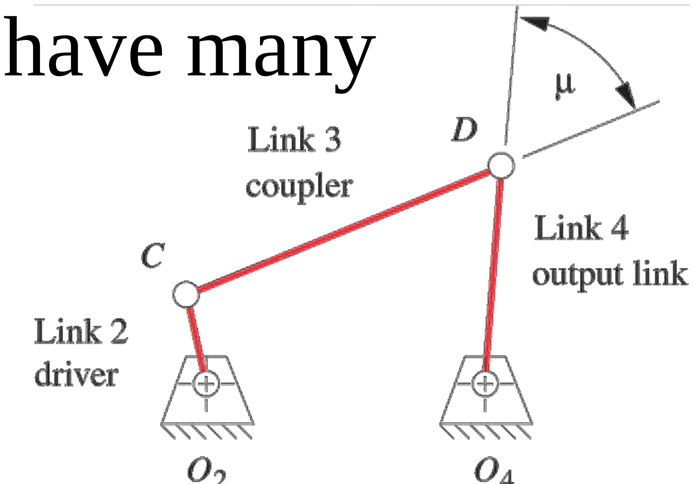
$$\mathbf{R}_{UO_4} = ue^{j(\theta_4 + \delta_4)}$$

$$\mathbf{R}_P = \mathbf{R}_A + \mathbf{R}_{PA}$$

$$\mathbf{R}_{PA} = pe^{j(\theta_3 + \delta_3)} = p\left[\cos(\theta_3 + \delta_3) + j\sin(\theta_3 + \delta_3)\right]$$

Transmission Angles

- We will expand that definition here to represent the angle between any two links in a linkage, as a linkage can have many transmission angles.

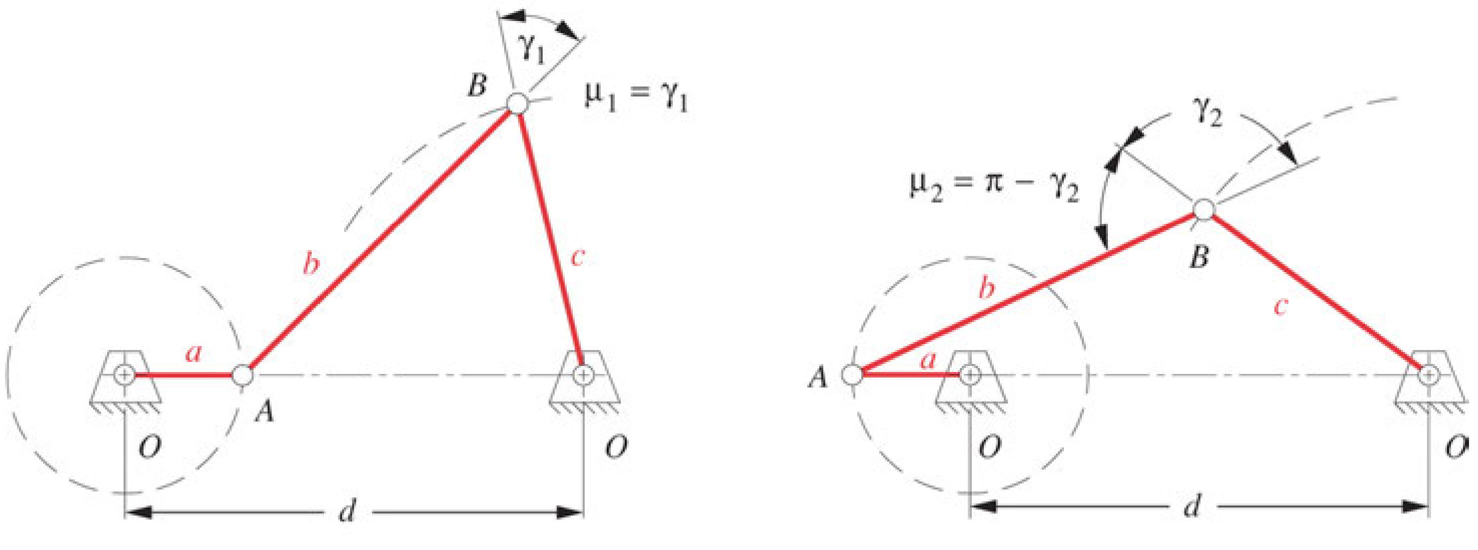


- It is easy to define the transmission angle algebraically. It is the difference between the angles of the two joined links.

Transmission Angles

Extreme Values of the Transmission Angle

- For a Grashof crank-rocker fourbar linkage the minimum value of the transmission angle occurs when the crank is colinear with the ground link



(a) Overlapped (b) Extended

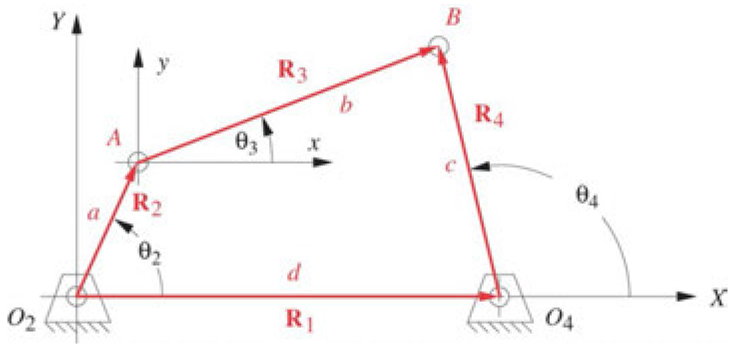
Transmission Angles

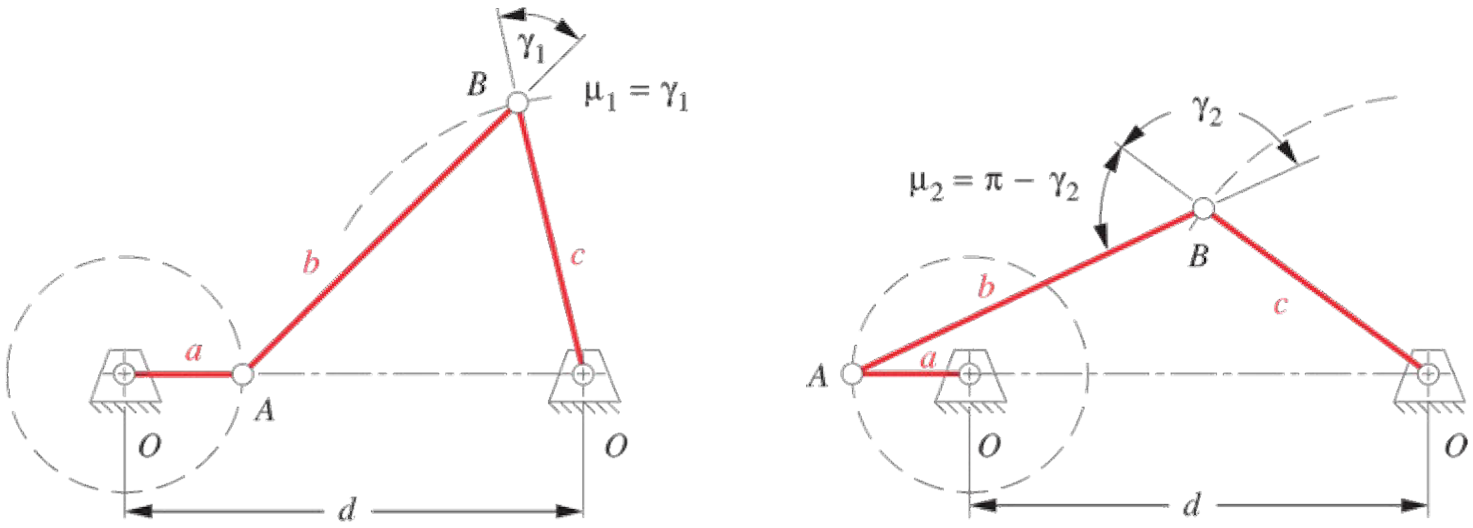
Extreme Values of the Transmission Angle

Grashof

$\theta 2=0, \pi$

$\mu=\theta 4-\theta 3$





(a) Overlapped

(b) Extended

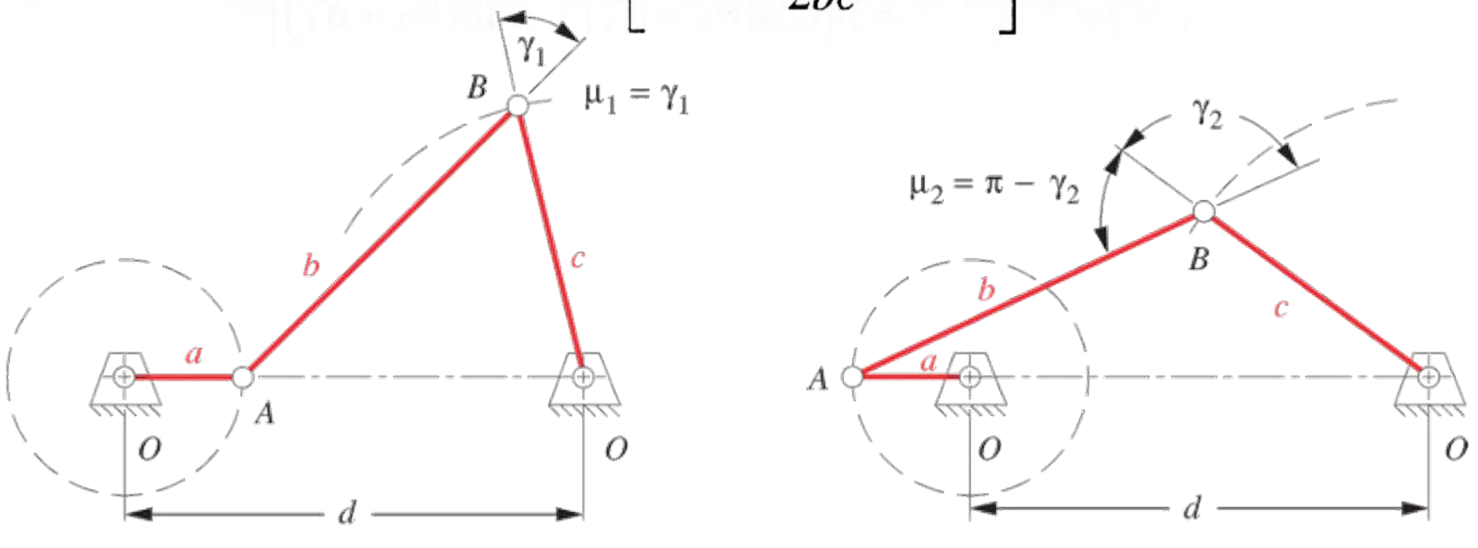
Transmission Angles

Extreme Values of the Transmission Angle

We label the links a = link 2; b = link 3; c = link 4; d = link 1

For the overlapping case (Figure 4-15a) the cosine law gives

$$\mu_1 = \gamma_1 = \arccos\left[\frac{b^2 + c^2 - (d-a)^2}{2bc}\right]$$



(a) Overlapped (b) Extended

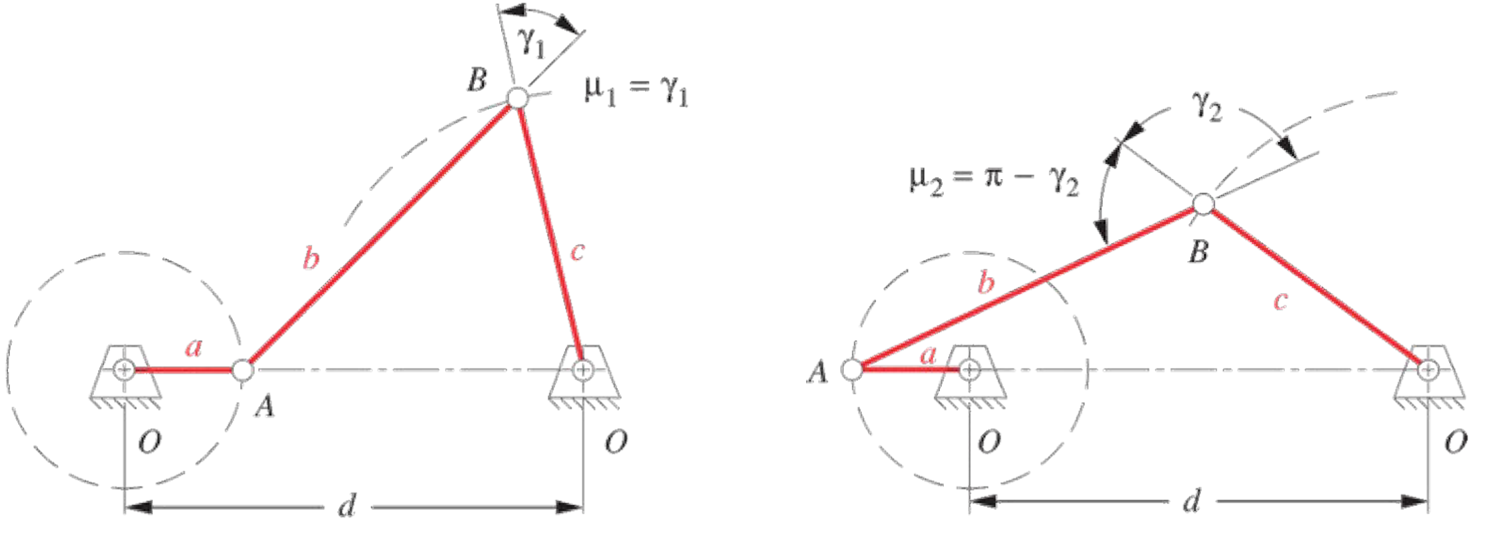
Transmission Angles

Extreme Values of the Transmission Angle

for the extended case, the cosine law gives

$$\mu_2 = \pi - \gamma_2 = \pi - \arccos\left[\frac{b^2 + c^2 - (d+a)^2}{2bc}\right]$$

- For a Grashof double-rocker linkage the transmission angle can vary from 0 to 90 degrees because the coupler can make a full revolution



(a) Overlapped (b) Extended

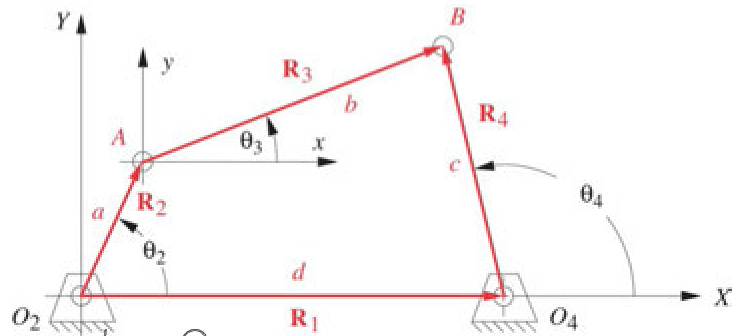
Transmission Angles

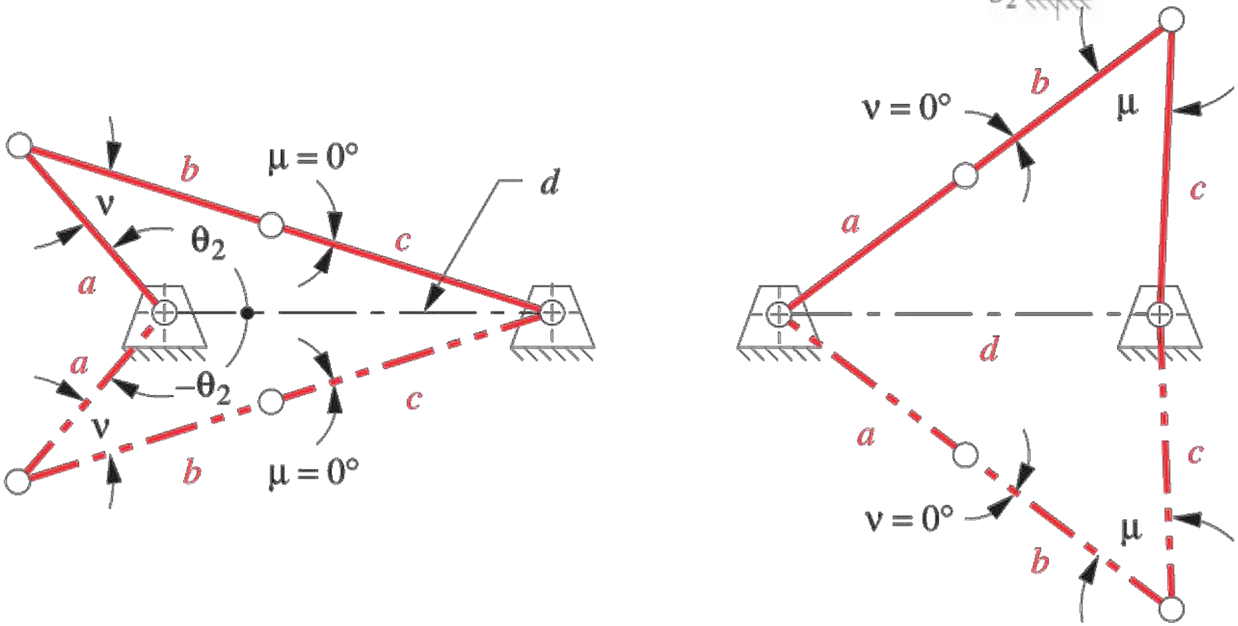
Extreme Values of the Transmission Angle

Non-Grashof

$\theta 2=\theta 3$

$\mu=\theta 4-\theta 3$





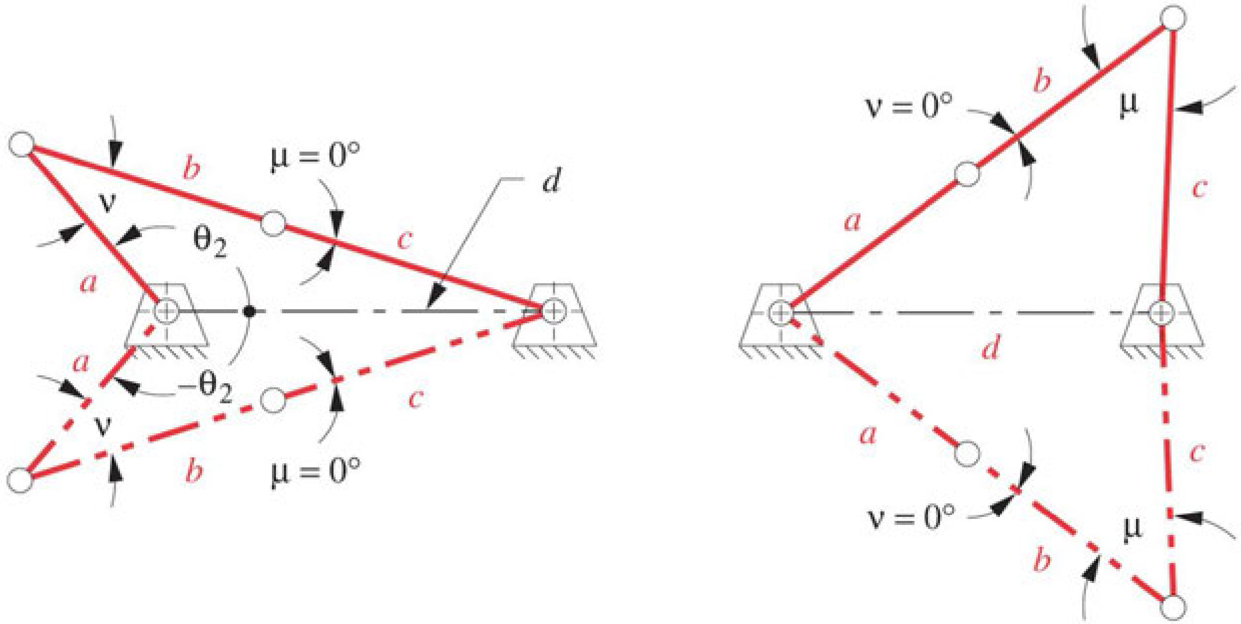
(a) Toggle positions for links *b* and *c*

(b) Toggle positions for links *a* and *b*

Transmission Angles

Extreme Values of the Transmission Angle

- For a non-Grashof triple-rocker linkage the transmission angle will be zero degrees in the toggle positions which occur when the output rocker c and the coupler *b* are colinear.



(*a*) Toggle positions for links *b* and *c*

(*b*) Toggle positions for links *a* and *b*

$$\nu = 0,$$

$$\mu = \arccos\left[\frac{(a+b)^2 + c^2 - d^2}{2c(a+b)}\right]$$

Not minimum μ