Chapter 2: single Random Variables and Probability Distribution.

Random Variable

Discrete

PMF : $P(X=x)$ — Probability mass Function

- Valid if:-
  $P(X=x) \geq 0$
  $\sum_{-\infty}^{\infty} P(X=x) = 1$

CDF : F_X — Comulative Distribution Function

- Defined as:
  $F_X(x) = P(X \leq x)$

Continuous

PdF : f_X — Probability density Function

- Valid if:
  $f_X(x) \geq 0$
  $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

CDF : F_X — Comulative Distribution Function

- Defined as:
  $F_X(x) = P(X \leq x)$

- **Properties of CDF :-**

شرح مبسط :-

1- $F_X(-\infty) = 0$

1- لأنه عند الـ $-\infty$ ما في احتمال (الـ $-\infty$ تبعي ولا الـ Function) يعني فش اشي وراها

2- $F_X(\infty) = 1$

2- لأنه عند الـ ∞ بدي احتمالية كل ما سبقها والي هو احتمالية كل اشي وبتساوي واحد

3- $0 \leq F_X(x) \leq 1$

3- لأنه الـ Function عبارة عن احتمال

4- $F_X(x_1) \leq F_X(x_2)$ if $x_1 \leq x_2$

4- يعني الاقتران متزايد

5- $F_X(x^+) = F_X(x)$

5- من اليمين

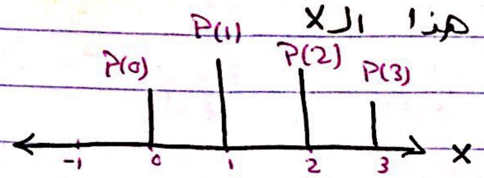
معناها أنا دخلت احتمالية X وما سبقها بالتالي هي نفسها $F_X(x)$

6- $P\{x_1 \leq X \leq x_2\} = F_X(x_2) - F_X(x_1)$

Discrete Random Variable :- Integer numbers Not intervals

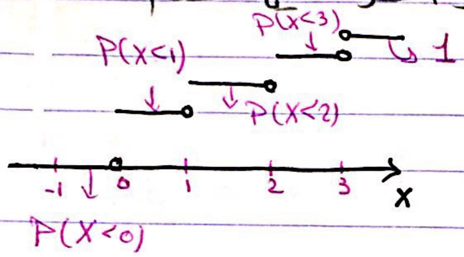
PMF:

• يكون عبارة عن قيم معينة عند كل X
وهو احتمالية حدوث هذا الـ X



CDF:

• يكون مجموع كل الاحتمالات
السابقة للاحتمال X وآخر احتمال لازم
يساوي واحد



Note:

→ In Discrete random Value :- * اشارة المساواة مهمة جداً وتحدث فرقاً في الإجابة

Ex: at $X = 2$

$P(X \leq 2) \neq P(X < 2)$

But at $X = 0.5$

$P(X \leq 0.5) = P(X < 0.5)$

→ You have to Know the differenc Between

$P(X = x)$ and $P(X \leq x)$ $(F_X(x))$

$P(X = x)$: This Gives you The probability at a value

$P(X \leq x)$: This Gives You The probability at the value and all Previous

I'll Explain more in the Example

Examples solved on Discrete Random value.

Example ①

► Given: $F_X(x) = \begin{cases} K & , x < -2 \\ 0.2 & , -2 \le x < 0 \\ G & , 0 \le x < 2 \\ 0.9 & , 2 \le x < 4 \\ H & , 4 \le x \end{cases}$ This is Cdf Given

$P(X \le x)$ is Given

Assuming $P(X \le 1) = 0.7$

→ Find K, G, H :-

$K = F(-\infty) = 0$

$H = F(\infty) = 1$

$P(X \le 1) = F_X(1) = G = 0.7$ • لما انو معطيك F_X فكانك بتعوض الفئة مباشرة

لو كان معطيك $P(X=x)$ بتصير $P(X \le 1)$ مجموع احتمالية الـ ١ وماقبلها

يعني

Example ②

► Given $P(X=x) = \begin{cases} 1/8 & , x = 0 \\ 3/8 & , x = 1 \\ 3/8 & , x = 2 \\ 1/8 & , x = 3 \end{cases}$ This is PMF Given

Find • $F_X(2) = P(X \le 2) = \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = \frac{7}{8}$

• $F_X(1^-) = \frac{1}{8}$ يعني احتمالية القيم اللي قبل الواحد بدونه

$- \; +$

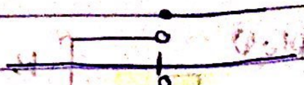
1

• $F_X(1^+) = \frac{3}{8} + \frac{1}{8} = \frac{4}{8}$

Scanned with CamScanner

In Example ① Determine PMF of this Cdf

at $X=0$:



$P(X=0) = P(X \geq 0) - P(X<0) = 0.7 - 0.2 = 0.5$

at $X=-2$

$P(X=-2) = P(X \geq -2) - P(X<-2) = 0.2 - 0 = 0.2$

and so on for $X = 2, 4$

$$PMF = \begin{cases} 0.2 , & X=-2 \\ 0.5 , & X=0 \\ 0.2 , & X=2 \\ 0.1 , & X=4 \\ 0 , & o.w \end{cases}$$

In Example ② Determine cdf for this PMF

$F_X(0) = 1/8$

$F_X(1) = P(X \leq 1) = P(X=1) + P(X<1) = \frac{3}{8} + \frac{1}{8} = 4/8$

$F_X(2) = P(X \leq 2) = P(X=2) + P(X<2) = \frac{3}{8} + \frac{4}{8} = 7/8$

$F_X(3) = P(X \leq 3) = P(X=3) + P(X<3) = \frac{7}{8} + \frac{1}{8} = 8/8$

$$F_X = \begin{cases} 1/8 & 0 \leq X < 1 \\ 4/8 & 1 \leq X < 2 \\ 7/8 & 2 \leq X < 3 \\ 8/8 & 3 \leq X \\ 0 & o.w \end{cases}$$

Mean and Variance of a Distribution.

▶ **Mean:** Discrete X :- $E\{\boxed{X}\} = \sum \boxed{X} P(X)$ → This can be any function $g(x)$

Continous X :- $E\{x\} = \int_{-\infty}^{\infty} x f_x(x)\, dx$

Ex. $E\{x^2\} = \sum x^2 P(x)$ (Dis)

$= \int_{-\infty}^{\infty} x^2 f_x(x)\, dx$ (Cont)

• Note: $E\{a\} = a$ where a is a constant

▶ **Variance:** $\sigma_x^2 = E\{(x - \mu_x)^2\}$ بقيس التوزيع

▶ **standard deviation:** $\sigma_x = \sqrt{(\sigma_x)^2}$

▶ **Median:** X_m

Only in Continous Random Value.

$P(X \le X_m) = 1/2$

$\int_{-\infty}^{X_m} f_x(x)\, dx = 1/2$

X_m should be within the interval.

Mean properties:

• $E\{ax\} = a\,E\{x\}$

• $E\{ax^2 + bx + c\}$

$= a\,E\{x^2\} + b\,E\{x\} + c$

▶ **Mode:** X_o

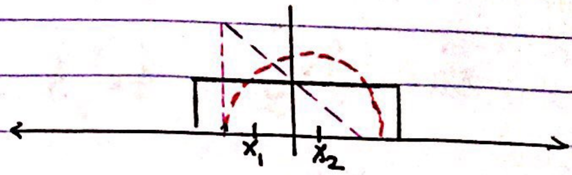
$f_x(X_o) = Max\ f_x(x)$

and so

we find $\frac{df_{x}(x)}{dx} = 0$ and take max value.

Continous Random value : real number Intervals

PDF



- It may have different shapes : Square , Parabola, triangle.

- Area represent probability.

$$P(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} f_x(x)\, dx$$

- The Whole Area = 1

- $$\int_{-\infty}^{\infty} f(x)\, d_x = P(-\infty < X < \infty) = 1$$

- You cannot find the probability at one value $P(X=K)$ is not valid.

  And so $P(K_1 \leq X \leq K_2) = P(K_1 < X < K_2)$

  Where K is a constant

CDF:

$$F_X(x) = \int_{-\infty}^{x} f_x(\tau)\, d\tau$$

$$f_x(x) = \frac{dF_x}{dx}$$

Common Discrete Random Variables:-

▸ The Binomial Distribution

Expirement repeated n times :-

$P(A) = P(X=k) = \binom{n}{k}[P(H)]^{k}[1-P(H)]^{n-k}$

where :

A : observing outcome H for K times

K :- number of times A happens.

Conditions

- two outcomes
- Prob of outcomes does change.
- trials are independant

In Binomial: $Mean = np(H)$

$\sigma_X^2 = n\,p(H)\,[1-p(H)]$

▸ The Geometric Distribution:-

Expirement Done X times to the first occurence of a success :-

$P(X=x) = (1-P)^{x-1}\,P$ where $X = 1, 2, 3, \ldots$

P: Probability of success

X: number of hits until success

In Geometric: $Mean = \frac{1}{p}$

$\sigma_X^2 = \frac{1-p}{p^2}$

Hypo-Geometric Distribution

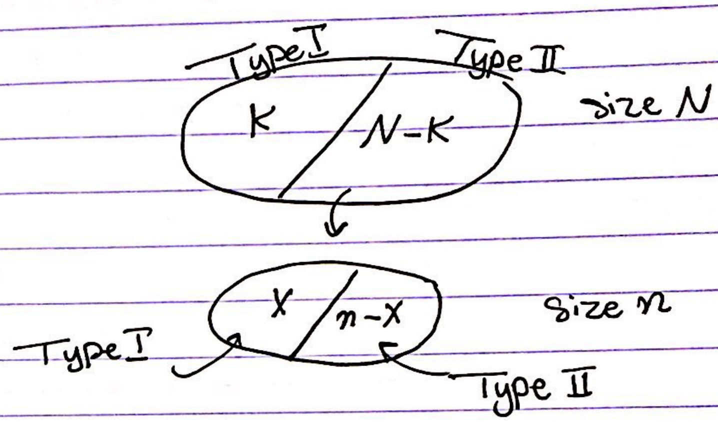
$$P(X=x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}} \qquad x = 0, 1, \dots \min(n, K)$$

- $p = \frac{K}{N}$ ratio of Items of Type ① of the whole pop

In Hypo-Geometric:

$\mu_x = np$

$\sigma_x^2 = np(1-p)\left(\frac{N-n}{N-1}\right)$



Poisson Distribution:-

$$P(X=x) = e^{-b}\,\frac{b^x}{x!} \qquad x = 0, 1, 2, \dots$$

$b > 0$

$b = \lambda T$ where λ: rate of occurency per time

In poissons:

$\mu_x = b$

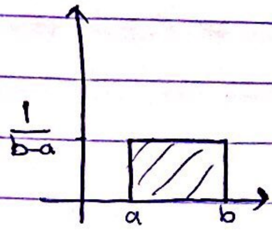
$\sigma_x^2 = b$

- Counting begins at $X(\ )=0$ (time $=0$)
- $p = \frac{b}{n}$

Common Continuous Distribution

▶ uniform distribution: $\rightarrow$ symmetry

$$f_X(x) = \begin{cases} \frac{1}{b-a} & , a \le x \le b \\ 0 & , o.w \end{cases}$$



$$\mu_X = \frac{b+a}{2}$$

$$\sigma_X^2 = \frac{b-a}{12}$$

$$F_X(x) = \begin{cases} 0 & , x < a \\ \frac{x-a}{b-a} & , a \le x < b \\ 1 & , b \le x \end{cases}$$

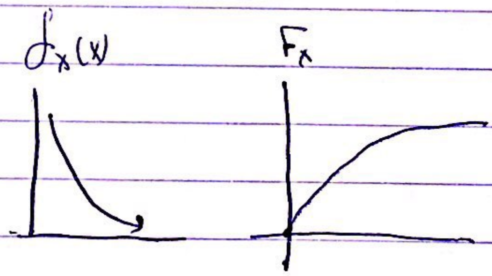
▶ Exponential Distribution:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & , x \ge 0 \\ 0 & , o.w \end{cases}$$

In Exponential:-

$$\mu_X = \frac{1}{\lambda}$$

$$\sigma_X^2 = \frac{1}{\lambda^2}$$



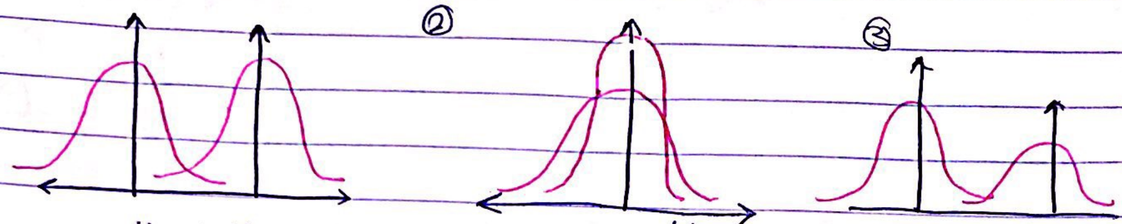
$$F_X(x) = \begin{cases} 0 & , x < 0 \\ 1 - e^{-\lambda x} & , 0 \le x \end{cases}$$

► Gaussien Distribution :-

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{\frac{-(x-\mu_x)^2}{2\sigma_x^2}} \qquad -\infty < x < \infty$$

► There are 3 cases :-

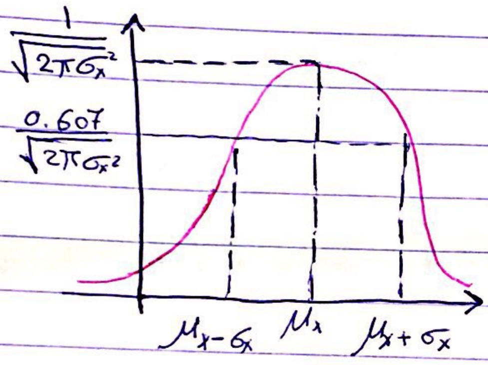
① ② ③



① $\mu_1 \neq \mu_2$, $\sigma_1 = \sigma_2$

② $\mu_1 = \mu_2$, $\sigma_1 \neq \sigma_2$

③ $\mu_1 \neq \mu_2$, $\sigma_1 \neq \sigma_2$



$$\Phi(z) = P(Z \leq z)$$

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{\frac{u^2}{2}} du$$

• Properties: $\Phi(z) = 1 - \Phi(-z)$

Now : → $P(X \leq x_0) = \Phi\left(\frac{x_0 - \mu_x}{\sigma_x}\right)$