جامعة بيرزيت

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Maximum Likelihood

Problem 1: Given a random sample of size (n) taken from a Gaussian population with parameters μ_x and σ_x^2 . Use the ML technique to find an unbiased estimator for the cases below:

a) The mean μ_x when the variance σ_x^2 is assumed known.

b) The mean μ_x when the variance σ_x^2 is assumed unknown.

c) The variance σ_x^2 when the mean μ_x is assumed known.

d) The variance σ_x^2 when the mean μ_x is assumed unknown.

Sample of size n $\rightarrow$ from a pop with Parameters μ_x and $\sigma_x{}^2$

a) $\mu_x \rightarrow \sigma_x{}^2$ known true variance

assuming r.vs $X_1, \ldots, X_n$ independant in the sample and Gaussien

approximation

$$f_{x_1 x_2 \ldots}(x_1, x_2 \ldots, x_n) = f_{x_1}(x_1) \ldots f_{x_n}(x_n)$$

since $X_1 \ldots X_n$ Ind

we want to estimat it

$$= \left[\frac{1}{\sqrt{2\pi\sigma_x{}^2}} e^{-\frac{(x_1-\hat{\mu}_x)^2}{2\sigma_x{}^2}}\right]$$

$$* \left[\frac{1}{\sqrt{2\pi\sigma_x{}^2}} e^{-\frac{(x_2-\hat{\mu}_x)^2}{2\sigma_x{}^2}}\right] \ldots$$

$$f_{x_1 \ldots x_n}(x_1 \ldots x_n) = \left(\frac{1}{\sqrt{2\pi\sigma_x{}^2}}\right)^n e^{-\frac{\sum(x_i-\mu_x)^2}{2\sigma_x{}^2}}$$

$$L(\hat{\mu}) = \frac{1}{\sqrt{2\pi\sigma_x{}^2}} e^{-\frac{\sum(x_i-\hat{\mu}_x)^2}{2\sigma_x{}^2}}$$

→ Now we either differentiate $L(\hat{\mu})$ or $\ln(L(\hat{\mu}))$, choose the easier since there are powers we use $\ln$ to make it easier.

$$\ln\left(L(\hat{\mu})\right) = \ln\left[\left(\frac{1}{2\pi\sigma_x^2}\right)^{-\frac{n}{2}} e^{\frac{-1}{2\sigma_x^2}\sum(x_i-\hat{\mu}_x)^2}\right]$$

$$= \ln\left(\frac{1}{2\pi\sigma_x^2}\right)^{-\frac{n}{2}} + \ln e^{\frac{-1}{2\sigma_x^2}\sum(x_i-\hat{\mu}_x)^2}$$

$$\ln\left(L(\hat{\mu})\right) = -\frac{n}{2}\ln\frac{1}{2\pi\sigma_x^2} - \frac{1}{2\sigma_x^2}\sum(x_i-\hat{\mu}_x)^2$$

Now Differentiate with respect to the parameter you want to estimate $(\hat{\mu}_x)$ and equalize it to zero

$$\frac{\partial \ln\left(L(\hat{\mu})\right)}{\partial \mu} = 0 - \frac{1}{2\sigma_x^2} 2\sum(x_i-\hat{\mu}_x)(-1) = 0$$

$$0 = \frac{\cancel{2}}{\cancel{2\sigma_x^2}} \sum x_i - \hat{\mu}_x$$

$$0 = \sum x_i - \sum \hat{\mu}_x \qquad \mu_x \text{ is constant}$$

$$\sum x_i = \sum \hat{\mu}_x$$

$$\sum x_i = \hat{\mu}_x \underbrace{\sum(1)}_{n}$$

so $\boxed{\hat{\mu}_x = \frac{\sum x_i}{n}}$

→ We need to make sure it's unbiased

$E\{\hat{\mu}_x\} = \mu_x$ means $\hat{\mu}_x$ is unbiased

$$E\left\{\frac{\sum x_i}{n}\right\} = \frac{1}{n} E\left\{X_1 + X_2 + \dots X_n\right\}$$

$$= \frac{1}{n}\left[E\{X_1\} + E\{X_2\} + \dots\right] = \frac{1}{n}\left[\mu_x + \mu_x \dots (\mu_x)_n\right]$$

$$= \frac{1}{n} n \, \mu_x = \mu_x \checkmark$$

b) $\hat{\mu}_x$ when σ_x^2 is unknown $(\hat{\sigma}_x^2)$

Same $L(\hat{\mu})$ But with $\sigma_x^2 \Rightarrow \hat{\sigma}_x^2$

$$\ln L(\hat{\mu}) = -\frac{n}{2} \ln \frac{1}{2\pi \hat{\sigma}_x^2} - \frac{1}{2\hat{\sigma}_x^2} \sum (x_i - \hat{\mu}_x)$$

$$\frac{\partial \ln L(\hat{\mu})}{\partial \hat{\mu}} = \cancel{\frac{1}{\hat{\sigma}_x^2}} \sum x_i - \hat{\mu}_x = 0$$

$$\boxed{\hat{\mu}_x = \frac{\sum x_i}{n}}$$ and It's unbiased as shown Before

c) σ_x^2 when μ_x is Known

$\hat{\sigma}_x^2$:-

Same : $\ln(L(\hat{\sigma}_x^2)) = \frac{-n}{2}\ln(2\pi\hat{\sigma}_x^2) - \frac{1}{2\hat{\sigma}_x^2}\sum(x_i - \underset{\text{Known}}{\mu_x})^2$

$$\frac{\partial \ln(L(\hat{\sigma}_x^2))}{\partial \hat{\sigma}_x^2} = -\frac{n}{2}\left(\frac{2\pi}{2\pi\hat{\sigma}_x^2}\right) - \frac{1}{2}\sum(x_i-\mu_x)^2(-1)(\hat{\sigma}_x^2)^{-2} = 0$$

$$\frac{-n}{2\hat{\sigma}_x^2} + \frac{\sum(x_i-\mu_x)^2}{(2)(\hat{\sigma}_x^2)^2} = 0$$

$$\cancel{\frac{1}{2\hat{\sigma}_x^2}}\left(-n + \frac{\sum(x_i-\mu_x)^2}{\hat{\sigma}_x^2}\right) = 0$$

$$\boxed{\hat{\sigma}_x^2 = \frac{\sum(x_i-\mu_x)^2}{n}}$$

Check for Biasing :-

$$E\{\hat{\sigma}_x^2\} = \sigma_x^2$$

$$E\left[\frac{\sum(x_i-\mu_x)^2}{n}\right] = \frac{1}{n}\sum E\left[(x_i-\mu_x)^2\right]$$

$$= \frac{1}{n}\sum\sigma_x^2 = \frac{1}{\cancel{n}}\cancel{n}\sigma_x^2 = \sigma_x^2$$

d) σ_x^2 when μ_x is unknown $(\hat{\mu}_x)$

Same $\frac{\partial \ln(\hat{\sigma}_x^2)}{\partial \hat{\sigma}_x^2} = 0$ But $(\mu_x \to \hat{\mu}_x)$

$$\frac{-n}{2\hat{\sigma}_x^2} + \frac{\sum(x_i - \hat{\mu}_x)^2}{2(\hat{\sigma}_x^2)^2} = 0$$

$$\hat{\sigma}_x^2 = \frac{\sum(x_i - \hat{\mu}_x)^2}{n}$$

Check for Biasing:

$$\boxed{E\{\hat{\sigma}_x^2\} = \sigma_x^2}$$

$$= \frac{1}{n} E\{\sum(x_i - \hat{\mu}_x)^2\}$$

$$= \frac{1}{n} E\{\sum(x_i - \hat{\mu}_x)(x_i - \hat{\mu}_x)\}$$

$$= \frac{1}{n} E\{\sum(x_i^2 - x_i\hat{\mu}_x - x_i\hat{\mu}_x + \hat{\mu}_x^2)\}$$

$$= \frac{1}{n} E\{\sum(x_i^2 - 2x_i\hat{\mu}_x + \hat{\mu}_x^2)\}$$

$$= \frac{1}{n} E\{\sum x_i^2 - 2\hat{\mu}_x \sum x_i + n\hat{\mu}_x^2\}$$

$$= \frac{1}{n} E\{\sum x_i^2 - 2n\hat{\mu}_x\hat{\mu}_x + n\hat{\mu}_x^2\}$$

$$= \frac{1}{n} E\{\sum x_i^2 - n\hat{\mu}_x^2\} = E\{\frac{\sum x_i^2}{n} - \hat{\mu}_x^2\}$$

We will be using two previous results

$$\sigma_x^2 = E\{X^2\} - (E\{X\})^2$$

$$E\{X^2\} = \sigma_x^2 + (E\{X\})^2 \quad \text{— ①}$$

and from Central limit Theorem

$$E\{\hat{\mu}_x\} = \mu_x \qquad Var\{\hat{\mu}_x\} = \frac{\sigma_x^2}{n}$$

$$\frac{\sigma_x^2}{n} = E\{\hat{\mu}_x^2\} - \mu_x^2$$

How? : $Var[\hat{\mu}_x] = E\{\hat{\mu}_x^2\} - (E\{\hat{\mu}_x\})^2$

$$\frac{\sigma_x^2}{n} = E\{\hat{\mu}_x^2\} - \mu_x^2 \rightarrow ②$$

Now putting in equations ① ②

$$\frac{1}{n} E\{\sum X_i^2 - n\hat{\mu}_x^2\} = \frac{1}{n}\left[\sum E\{X_i^2\} - nE\{\hat{\mu}_x^2\}\right]$$

$$= \frac{1}{n}\left[\sum\left(\sigma_x^2 + (E\{X\})^2\right) - n\left(\frac{\sigma_x^2}{n} + \mu_x^2\right)\right]$$

$$= \frac{1}{n}\left[n\sigma_x^2 + n\mu_x^2 - \sigma_x^2 - n\mu_x^2\right]$$

$$= \sigma_x^2 + \cancel{\mu_x^2} - \frac{\sigma_x^2}{n} - \cancel{\mu_x^2}$$

$$= \sigma_x^2\left(\frac{n\cancel{x}}{1\times n} - \frac{1}{n}\right) = \sigma_x^2 \left(\frac{n-1}{n}\right)$$

we need to get rid of it

$$= \sigma_x^2 \cancel{\left(\frac{n-1}{n}\right)\left(\frac{n}{n-1}\right)}^{1}$$

so $\hat{\sigma}_x^2 = \sum \frac{(X_i - \hat{\mu}_x)^2}{\cancel{n}}\left(\frac{\cancel{n}}{n-1}\right) = \frac{\sum(X_i - \hat{\mu}_x)^2}{n-1}$

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